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Duality Principle and New Forms of the Inverse Laplace Transform for Signal Propagation Analysis in Inhomogeneous Media with Dispersion

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Abstract—New equations for Laplace transform inversion are obtained. The equations satisfy the causality principle. The impulse response of a channel is determined in order to analyze dispersion distortions in inhomogeneous media. The impulse response excludes the possibility that the signal exceeds the speed of light in the medium. The transmission bandwidth, the angular spectrum, and the Doppler shift in the ionosphere are computed.

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1. The solution of fundamental problems in radio engineering, radio physics, electrodynamics, remote sensing, computer science, electrical engineering, and other fields is based on the Laplace transform [1]. The frequency response to monochromatic signal transmission is usually found in the theory [2]. A promising approach relies on the impulse function as applied to analyzing time distortions of signals [3].

The goal of this paper is to use the duality principle [4–6] and the parametric method of analytic continuation to derive a new Laplace-transform equation and to determine an impulse response that satisfies the causality principle and rules out the possibility of exceeding the speed of light in free space. The image of the impulse response is the frequency response of the dispersive medium to a monochromatic signal.

The Laplace transform $V(s)$ of a real function of time, $f(t)$, has the form

$$V(s) = \int_0^\infty \exp(-st) f(t) dt; \; f(t) = 0, \quad t < 0. \tag{1}$$

Relations that hold for $t > 0$ were obtained in [4–6], namely,

$$f_1(t) = \frac{i[V_c(it) - V_c(-it)]}{\pi}; \quad f_2(t) = \frac{[V_s(it) + V_s(-it)]}{\pi}, \tag{2}$$

where i is the imaginary unit and $V_c(x)$, $V_s(x)$ are the cosine and sine Fourier transforms of the image function $V(s)$ on the real axis of the variable s:

$$V_c(x) = \int_0^\infty V(s)\cos sx\, ds; \quad V_s(x) = \int_0^\infty V(s)\sin sx\, ds. \tag{3}$$

The original function $f(t)$ for the image $V(s)$ exists for $t > 0$ if the functions $f_{1,2}(t)$ determined by Eqs. (2) coincide:

$$f(t) = f_1(t) = f_2(t), \quad t > 0. \tag{4}$$

In (3) we introduce a real parameter l by making the substitutions $x = lt$ and $s = \frac{\omega}{l}$:

$$V_c(lt) = l^{-1}\int_0^\infty V(l^{-1}\omega)\cos t\omega\, d\omega; \quad V_s(lt) = l^{-1}\int_0^\infty V(l^{-1}\omega)\sin t\omega\, d\omega. \tag{5}$$

The imaginary parameter values $l = \pm i$ are substituted into (5). Assuming that there exists a well-defined analytic continuation $V(\pm i\omega)$, we find from (2) and (5) that

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$$f_1(t) = \pi^{-1}\int_0^\infty V_+ \cos t\omega \, d\omega;$$
$$f_2(t) = i\pi^{-1}\int_0^\infty V_- \sin t\omega \, d\omega; \quad V_\pm = V(i\omega) \pm V(-i\omega). \tag{6}$$

The original function $f(t)$ in (6) is represented by the cosine/sine Fourier transforms of the even function $V_+ = V(i\omega) + V(-i\omega)$ and the odd function $V_- = V(i\omega) - V(-i\omega)$ of frequency ω, respectively.

Substituting $V(\pm i\omega)$ given by (1) into (6) and assuming that the order of integration can be changed, we obtain

$$f_{1,2}(t) = \pi^{-1}\int_0^\infty f(x)dx\int_0^\infty \{\cos[\omega(t-x)] \pm \cos[\omega(t+x)]\}d\omega$$
$$= \int_0^\infty f(x)[\delta(t-x) \pm \delta(t+x)]dx, \tag{7}$$

where $\delta(x)$ is the delta function and the upper and lower signs before the second term on the right-hand side of (7) correspond to the functions $f_1(t)$ and $f_2(t)$ in (2). According to (4) and (7), for $t \geq 0$, the functions $V(s)$ are associated with a unique original function $f(t)$.

Because of the influence exerted by the terms $\pm\delta(t+x)$, for $t < 0$, the functions $f_{1,2}(t)$ in Eq. (7) do not vanish, which contradicts the causality principle. The duality principle (2) makes it possible to satisfy the causality condition $f(t) \equiv 0$, $t < 0$, by specifying $f(t)$ in the form $f(t) = \frac{[f_1(t) + f_2(t)]}{2}$. From (2), (6), and (7), we derive the following relations for the inverse Laplace transform:

$$f(t) = \frac{[f_1(t) + f_2(t)]}{2}$$
$$= (2\pi)^{-1}[V_s(it) + V_s(-it) + iV_c(it) - iV_c(-it)], \tag{8}$$
$$t > 0; \quad f(t) \equiv 0, \quad t < 0;$$

$$f(t) = (2\pi)^{-1}\int_0^\infty (V_+ \cos t\omega + iV_- \sin t\omega)\, d\omega$$
$$= (4\pi)^{-1}\int_{-\infty}^\infty (V_+ \cos t\omega + iV_- \sin t\omega)\, d\omega, \tag{9}$$

where the functions $V_\pm$ are defined in (6). In the lower limit of integral (9), zero was replaced by $-\infty$, since the integrand is an even function.

An inversion formula is derived from (9) by using the bilateral Fourier transform (provided that V_+ and V_- are an even and an odd function of ω, respectively):

$$f(t) = (2\pi)^{-1}\int_{-\infty}^\infty V(\pm i\omega)\exp(\pm it\omega)\, d\omega; \tag{10}$$
$$f(t) = 0, \quad t < 0.$$

The image $V(\pm i\omega)$ is recovered from (10) by integrating the left- and right-hand sides with infinite limits and the kernel $\exp(\mp itx)$ and taking into account the equality $f(t) = 0, t < 0$:

$$V(\pm ix) = \int_0^\infty f(t)\exp(\mp itx)\, dt$$
$$= (2\pi)^{-1}\int_{-\infty}^\infty V(\pm i\omega)\exp(\pm it\omega \mp itx)\, d\omega \tag{11}$$
$$= \int_{-\infty}^\infty V(\pm i\omega)\delta(\omega - x)\, d\omega.$$

Equations (1), (9) and (1), (10) are pairs of transforms from the original to the image and, on the contrary, from the image to the original.

2. For a monochromatic signal in the case of a time dependence, the frequency transfer characteristic of a radio channel is defined as [5]

$$V_{GL}(\pm i\omega) = A(n)\exp(\mp i\omega c^{-1}\Phi);$$
$$\Phi(n) = \int_G^L n[\omega, l(\omega)]\, dl; \tag{12}$$
$$A(n) = A_0[X(n)\Gamma(n)]^{1/2},$$

where $n[\omega, l(\omega)]$ is the refractive index, dl is the element of the ray trajectory GL, $\Phi(n)$ is the phase delay on the ray GL, A_0 is the radio wave amplitude at the point L in the absence of medium inhomogeneities, $A(n)$ is the amplitude taking into account the radio wave attenuation caused by refraction $X(n)$ and absorption $\Gamma(n)$, and c is the speed of electromagnetic waves in free space. The integral in (12) is taken along the ray GTL (Fig. 1a). The position of GTL in the medium depends on the radio wave frequency ω. The refractive index $n(\omega)$ and, hence, Φ are even functions of ω. Below, we find the original $h(t, L)$ of the frequency channel transmission function $\exp(\pm i\omega\Phi)$.

Assume that the medium is spherically symmetric with the center at the point O. The target parameter p is constant along the trajectory GTL in the plane GOL (p is the length of the perpendicular drawn from the center O to the tangent at the current point of GTL). For fixed θ, R_L, R_G, the function $\Phi(n)$ satisfies the relations

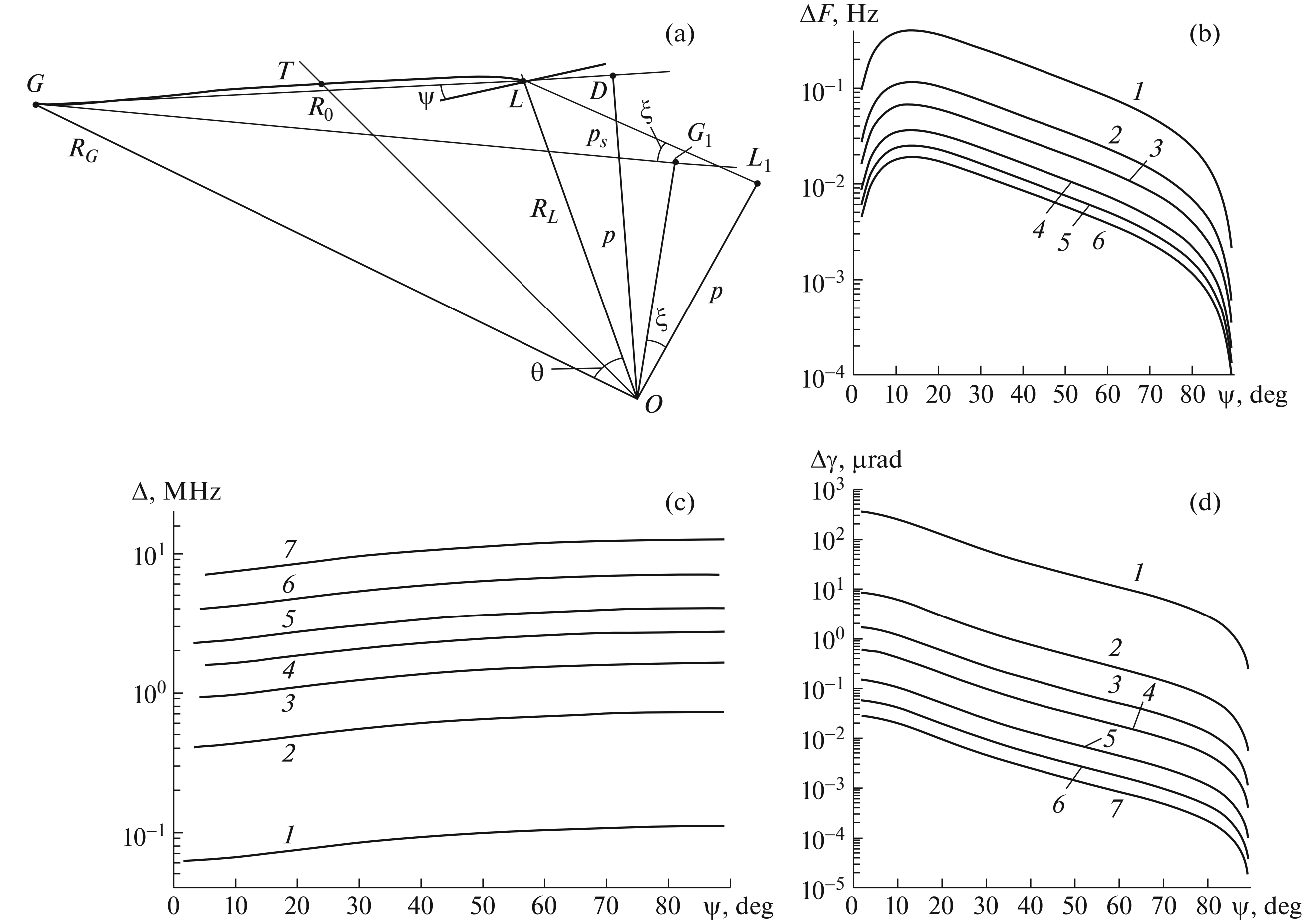

Fig. 1. (a) Schematic view of radio waves propagation through a disperse spherically symmetric layered inhomogeneous medium for a communication line consisting of a navigation satellite G and a receiver L on the Earth. The center of symmetry is at the point O. The line segment indicates the position of the horizontal line near the receiver L. The parameters (b) ΔF (Hz), (c) Δ (MHz), and (d) $\Delta\gamma$ (μrad) computed as functions of the site angle ψ and the frequency f for a signal transmitted from a satellite moving in a circular orbit at an altitude of 20 000 km. Curves *1–7* correspond to the frequency f equal to 60, 210, 360, 660, 810, 1110, and 1410 MHz, respectively.

$$\Phi(n) = \int_{R_L}^{R_G} \frac{(n^2r^2 - p^2)^{1/2}dr}{r} + p\theta;$$
$$\theta = p\int_{R_L}^{R_G} \frac{dr}{r(n^2r^2 - p^2)^{1/2}}; \quad \Phi'_\omega(n) = \int_G^L n'_\omega dl, \tag{13}$$

where θ is the central angle with vertex at the point O between the directions OG and OL and R_L, R_G, r are the distances OG, OL, and OT. The derivative p'_ω is not involved in (13), since the differentiation of the first equation in (13) yields $\Phi'_p(n) = 0$ for fixed θ, R_L, R_G. Let us find the impulse function $h(t, L)$ relating the response $f(t, L)$ at the output L to the signal $f(t)$ at the input of the channel G in the Duhamel integral:

$$f(t, L) = \int_0^t f(\tau)h(t - \tau, L)\, d\tau. \tag{14}$$

The function $h(t, L)$ is represented in the form

$$h(t, L) = \kappa\int_{-\infty}^{\infty} \cos[\omega(t - \Phi)]\, d\omega, \tag{15}$$

where $\kappa = \pi^{-1}$ and $0.5\pi^{-1}$ in the presence and absence of dispersion, respectively. In the absence of dispersion, n, $A(n)$, and Φ in (12) and (15) do not depend on frequency and $h(t, L)$ is the delta function: $h(t, L) = \delta(t - \Phi)$. The medium does not introduce and distortions, and the signal $f(t, L)$ (14) at the receiver input L preserves the emitted form $f(t)$ with a delay equal to Φ.

In the case of dispersion, low frequencies are absent because of the influence of the ionosphere. Substituting $h(t, L)$ given by (15) into (1) as $s \to is$, $f(t) \to h(t, L)$, changing the order of integration with respect to t, ω, and making the substitutions

$\omega = s - y, \omega = y - s$, we obtain two Hilbert transforms with the singular point $y = 0$:

$$V_{GL}(is) = (2\pi)^{-1}\int_{-\infty}^{\infty} dyy^{-1}\exp(-is\Phi)\exp[iy(\Phi+\omega\Phi'_{\omega})]i^{-1}$$
$$= \pi^{-1}\exp(-is\Phi) \qquad (16)$$
$$\times\int_{-\infty}^{\infty} dyy^{-1}\sin[y(\Phi + s\Phi'_{s})] = \exp(-is\Phi),$$

i.e., it follows from (1) that the frequency transfer function (12) $\exp(-is\Phi)$ is the image of the original function (15) $h(t, L)$. Substituting $h(t, L)$ given by (15) into the Duhamel integral, assuming that, at the input to the ionosphere, the function has the form of a damped sinusoid $f(\tau) = \exp(-\alpha\tau)\cos(\omega_0\tau)$, and changing the order of integration, in a similar manner, we find the signal in the ionosphere: $f(t) = \exp[-\alpha t - (\Phi + \omega\Phi'_{\omega})]\cos[\omega_0(t - \Phi)]$. This signal propagates with phase velocity along the phase path and with group velocity along the envelope. The first derivative of $h(t, L)$ determines the distance to the given point in the ionosphere, while the other derivatives determine the shape of the dispersion line at this point. This can be shown using the dispersion equation for radio waves in the ionosphere: $n^2(\omega) = 1 - \omega_p^2\omega^{-2}$, where $\omega_p = \alpha_0 N_e^{1/2}$ is the plasma frequency expressed in radians, α_0 is a dimensional coefficient, and N_e is the concentration of electrons. At the stationary point ω_s of integral (16), the equation $\varphi'_{\omega}(\omega, t) = [\omega(t - c^{-1}\Phi]'_{\omega} = 0$ holds, which has the form

$$t - c^{-1}R_0 = c^{-1}(d_g - R_0); \quad d_g = \Phi + \omega\Phi'_{\omega};$$
$$\varphi''_{\omega^2}(\omega, t) = c^{-1}d'_{g\omega}; \quad d'_{g\omega} = 2\Phi'_{\omega} + \omega\Phi''_{\omega^2}, \qquad (17)$$

where R_0 and d_g are the length of the interval GL and the spatial delay of radio waves along the path GTL at the frequency ω_s, respectively. The quantity d_g depends on the phase path Φ and its frequency derivative Φ'_{ω}. Importantly, for the chosen dispersion law, $\beta(\omega)$ determined by the relations

$$d_g - R_0 = \omega^{-2}\beta(\omega);$$
$$\beta = \int_G^L \omega_p^2 n^{-1}(n+1)^{-1}\,dl + \left(\int_G^L dl - R_0\right)\omega^2; \qquad (18)$$
$$\omega^2 = \beta(\omega)c^{-1}(t - c^{-1}R_0)^{-1},$$

is positive. Therefore, the first equation in (17) has no solutions for $t \le c^{-1}R_0$. The signal travel time along the path GTL is always longer than the time of signal propagation in free space along the line GL.

The first equation in (17) determines the argument t of $h(t, L)$ depending on the frequency at the stationary point ω_s. The spectral components of the signal with different frequencies arrive at the observation point L at different times. The derivative of the argument t with respect to frequency is given by the relation $t'_{\omega} = c^{-1}d'_{g\omega}$. According to (17), the delay derivative $d'_{g\omega}$ depends on the first and second derivatives of the phase path with respect to frequency. According to (17), the phase of the frequency transfer function $\varphi(\omega, t) = \omega(t - c^{-1}\Phi)$ near the stationary point ω_s of integral (16) is

$$\varphi(\omega, t) = \omega_s(t - c^{-1}\Phi_s) + \frac{(\omega - \omega_s)^2}{2\Delta^2};$$
$$\Delta^2 = \frac{c}{d'_{g\omega}(\omega_s, t)}. \qquad (19)$$

The quantity $\Delta = [c^{-1}d'_{g\omega}(\omega_s, t)]^{-1/2}$ in (19) is physically interpreted as the transmission bandwidth of the ionosphere at the given frequency ω_s. If the bandwidth of modulated radio waves $\Delta_W = |\omega - \omega_0|$ satisfies the condition $\Delta_W \ge \Delta$, then the signal is substantially distorted by the ionosphere. Under the reverse inequality $\Delta_W \le \Delta$, according to [2], all the derivatives of the phase $\varphi(\omega, t)$ in (17), except for the first one, can be neglected and $\varphi(\omega, t)$ can be represented in the form $\varphi(\omega_0, t) + \varphi'_{\omega}(\omega_0, t)(\omega - \omega_0)$, where ω_0 is the carrier frequency of the narrow-band signal. In this case, the signal is transmitted nearly without distortions, and the integration with respect to frequency in (17) gives the dependence of $h(t, L)$ on time in the form of a delta function:

$$h(t, L) = A(n)\delta(t - t_g); \quad t_g = v_g^{-1}d_g(\omega_0);$$
$$v_g(\omega_0) = \frac{cR_0}{d_g(\omega_0)} = cR_0(\Phi + \omega_0\Phi'_{\omega})^{-1}, \qquad (20)$$

where t_g and v_g are the group time delay and the group velocity of the narrow-band pulse at the frequency ω_0. The parameters t_g and v_g depend on ω_0.

For a rigorous analysis of the ionospheric distortions introduced into the signal, the function $h(t, L)$ has to be substituted into the Duhamel integral (14) in integral form (16).

3. The eikonal $\Phi(p, \omega)$ in (12) depends on the target parameter $p = R_{nL}R_{nG}D^{-1}\sin(\theta - \xi)$, which is constant on the trajectory of radio wave propagation, i.e., on the ray GTL in the plane GOL, and is determined by the relations [5]

$$\Phi(p,\omega) = L(p) + \kappa(p); \quad L(p) = D + p\xi(p);$$

$$\xi(p) = \sin^{-1}\frac{p_s}{R_L} - \sin^{-1}\frac{p}{R_{nL}} + \sin^{-1}\frac{p}{R_{nG}} - \sin^{-1}\frac{p_s}{R_G},$$

$$\xi = -\kappa'_p; \quad \kappa(p) = -\int_{R_L}^{R_G} n'_r(r,\omega)\frac{(n^2r^2 - p^2)^{1/2}dr}{n}; \tag{21}$$

$$D = \sqrt{R_{nG}^2 - p^2} - \sqrt{R_{nL}^2 - p^2}; \quad R_{nL,G} = n(R_{L,G})R_{L,G},$$

where $n(R_L)$ and $n(R_G)$ are the refractive indices at the points G and L; $\xi(p)$ is the angle of refraction of radio waves on the path GTL; R_L, R_G, and r_T are the distances OL, OG, and OT; and p_s is the target parameter of the line GL with respect to the center of symmetry O (Fig. 1a). In Eq. (21), $L(p)$ is equal to the sum of the length $p\xi(p)$ of the circular arc L_1G_1 of radius p and the differences of the tangent segments LL_1, GG_1 to the ray GTL drawn at the points L, G. The refractive parameter $\kappa(p,\omega)$ in (21) depends on the vertical gradient of the refractive index $n'_r(r,\omega)$ and describes the difference of the phase path $\Phi(p,\omega)$ from the length $L(p)$. For fixed values of the frequency ω and the parameters R_L, R_G, θ, p_s, the phase path $\Phi(p,\omega)$ in (13) satisfies the equation $\frac{\partial\Phi(p,\omega)}{\partial p} = 0$, which corresponds to Fermat's principle.

The harmonic components of the signal propagate through the medium along different trajectories, which are associated with correspond different values of the target parameter p. This causes a broadening $\Delta\gamma_L(\omega)$ of the angular spectrum of radio waves and introduces distortions into the signal. The broadening $\Delta\gamma_L(\omega)$ in the plane GOL at the receiver (transmitter) $L(G)$ is determined for $n(R_G) = n(R_L) = 1$ and the frequency band $\Delta\omega$:

$$\Delta\gamma_{L,G}(\omega) = \gamma'_{\omega L,G}(\omega)\Delta\omega = (R_{L,G}^2 - p^2)^{-1/2}p'_\omega(\omega)\Delta\omega, \tag{22}$$

$$p'_\omega(\omega) = \left[(R_G^2 - p^2)^{-1/2} - (R_L^2 - p^2)^{-1/2} - \frac{\partial\xi(p,\omega)}{\partial p}\right]^{-1} \times\frac{\partial\xi(p,\omega)}{\partial\omega}; \tag{23}$$

$$\frac{\partial\xi(p,\omega)}{\partial\omega} = -\frac{\partial^2\kappa(p,\omega)}{\partial p\partial\omega}.$$

In (22) and (23), $\gamma_{L,G}(\omega)$ is the angle with vertex at the point $L(G)$ between the direction toward the center O of spherical symmetry and the tangent to the trajectory at the point $L(G)$ (Fig. 1a). The width of the angular spectrum is reduced away from the center of symmetry.

Figures 1b–1d show the computed characteristics of ionospheric radio waves on the satellite–Earth path as functions of the site angle ψ and the frequency f (the satellite moves in a circular orbit at an altitude of 20000km). The electron density $N_e(h)$ was determined using the Chapman formula

$$N_e(h) = N_0\exp\{0.5[1 - 2(h - h_1)/d_1 - \sec\chi\exp(-2(h - h_1)/d_1)]\}$$

with $N_0 = 10^6$ el/cm^3, $h_1 = 300$ km, $d_1 = 100$ km, and $\chi = 0$, where N_0 is the electron density in the layer maximum, h_1 and d_1 are the height and the thickness of the layer, and χ the Sun's zenith angle.

Figure 1b presents the maximum correction to the Doppler frequency $\Delta F = \lambda^{-1}(p - p_s)d\theta/dt$ expressed in Hz in the daytime ionosphere. At a frequency of 400 MHz, this quantity is at most 0.6 Hz. Figure 1c shows the ionospheric transmission bandwidth Δ in MHz. The value of Δ is doubled as the angle ψ increases from 1° to 89°. As the frequency increases, Δ grows according the formula $\Delta \approx C_\Delta\omega^{3/2}$, where the parameter C_Δ is inversely proportional to the integral electron density on the path GTL.

Figure 1d shows the width $\Delta\gamma$ of the angular spectrum of radio waves (in microradians) for a signal bandwidth of 1 MHz. The quantity $\Delta\gamma$ exhibits a pronounced dependence on ψ. At a frequency of 400 MHz, $\Delta\gamma$ reaches 10 μrad at $\psi = 1°$ and then decreases by three orders of magnitude at $\psi = 89°$.

4. The results obtained above are of theoretical and practical importance. Relying on them, the impulse transfer function can be used to compute signal propagation through dispersive media, in lines of radio communication, radio navigation, and radio control, and in other information channels. Dependences of the transmission bandwidth, Doppler shift, and the width of the radio wave spectrum on the navigation satellite–Earth path were determined. Two new equations for Laplace transform inversion were obtained using the duality principle and the parametric method of analytic continuation. The inverse transform satisfies the causality principle. The impulse response of a dispersive medium shows that the signal velocity cannot exceed the speed of light.

The features of the obtained transforms as applied to modulated signals propagating through inhomogeneous media with various types of dispersion will be studied in a future publication.

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