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Oscillations and Synchronization in a System of Three Reactively Coupled Oscillators

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We consider a system of three interacting van der Pol oscillators with reactive coupling. Phase equations are derived, using proper order of expansion over the coupling parameter. The dynamics of the system is studied by means of the bifurcation analysis and with the method of Lyapunov exponent charts. Essential and physically meaningful features of the reactive coupling are discussed.

Keywords: Synchronization; quasi-periodic oscillation; bifurcation; chaos.

1. Introduction

The phenomena of synchronization of oscillators are wide spread in nature and technology. A variety of examples can be found in electronics, laser physics, biophysics, chemistry, neuroscience, etc. [Pikovsky *et al.*, 2001; Landa, 1996; Balanov *et al.*, 2009; Kuramoto, 1984; Glass & Mackey, 1988; Winfree, 2001]. The synchronization in modern experiments for optomechanical, micromechanical, and electronic oscillators is investigated, e.g. see [Heinrich *et al.*, 2011; Zhang *et al.*, 2012; Temirbayev *et al.*, 2013; Martens *et al.*, 2013; Tinsley *et al.*, 2012]. The general picture of oscillatory modes in arrays of elementary oscillators substantially depends on the number of elements and of the coupling type. In the simplest case of two oscillators with *dissipative coupling*, one can observe mutual mode-locking of the oscillators with different natural frequencies, two-frequency quasi-periodic oscillations, the effect of "the oscillation death", and the regime called the "broadband synchronization" specific for the oscillatory elements with nonidentical control parameters [Pikovsky *et al.*, 2001; Landa, 1996; Balanov *et al.*, 2009; Aronson *et al.*, 1990;

Rand & Holmes, 1980; Ivanchenko *et al.*, 2004; Kuznetsov *et al.*, 2009; Kuznetsov & Roman, 2009]. With increase of the number of oscillatory elements, the picture becomes more complicated; many of its features have been established and understood recently [Ashwin *et al.*, 1993; Ashwin, 1998; Linsay & Cumming, 1989; Ashwin *et al.*, 2008; Maistrenko *et al.*, 2004; Emelianova *et al.*, 2013; Emelianova *et al.*, 2014].

More complex is a case of *reactive coupling* (termed sometimes as the conservative coupling) [Pikovsky *et al.*, 2001; Landa, 1996; Balanov *et al.*, 2009; Rand & Holmes, 1980; Ivanchenko *et al.*, 2004; Kuznetsov *et al.*, 2009]. In radio-engineering and electronics, the coupling of such kind occurs in the presence of a reactive element (inductance) in the coupling circuit instead of a resistor giving rise to dissipative coupling [Landa, 1996]. A typical example of a system with reactive coupling corresponds to ionic traps [Lee & Cross, 2011]. In such traps, ions are confined using variable microwave fields, which restrict the magnitude of radial oscillations of the ions, and a constant electric field, limiting the axial motions. In a trap with many electrodes, the ions form a chain located in the potential wells, and the nonlinearity provides the anharmonic nature of oscillations of the ions in the wells. Additionally, the ions are irradiated by laser beams. The blue laser light of frequency larger than the natural frequency of the ion oscillations provides instability in the system. The red laser light of frequency less than the oscillation frequency gives rise to dissipation. The ions in the chain are coupled due to the Coulomb repulsion. The simplest model of such a system is a chain of reactively coupled van der Pol oscillators [Lee & Cross, 2011]. Another study focuses on the models of coupled van der Pol oscillators in the application for biological circadian rhythms [Rompala *et al.*, 2007] and for arrays of nanoscale mechanical resonators [Cross *et al.*, 2004].

The reactive coupling is a phenomenon essentially more subtle than the dissipative coupling. The reason is that when constructing a reduced phase model one must take into account the second order effects in the coupling magnitude. In the case of two oscillators in the first order approximation, the coupling effects are generally not manifested. For three or more oscillators the linear terms are present, but the resulting abridged system is conservative [Pikovsky & Rosenau, 2006; Rosenau & Pikovsky, 2005].

The case of two oscillators was discussed in [Rand & Holmes, 1980; Ivanchenko *et al.*, 2004; Kuznetsov *et al.*, 2009]. There the appropriate model for the dynamics of the phase variable was derived and it was shown that the synchronization effect appears only when taking into account the terms of the second order in the coupling parameter. One more feature of the dynamics with the reactive coupling is the phase bistability; it means that depending on initial conditions, the oscillators may synchronize either in-phase, or in counter-phase. In the present paper, we consider the dynamics and synchronization in a chain of three reactively coupled van der Pol oscillators. In contrast to [Pikovsky & Rosenau, 2006; Rosenau & Pikovsky, 2005], we will assume nonidentical oscillators for frequency and focus on the discussion of the structure of parameter plane of frequency detuning. The first task is to derive the correct phase equations accounting for all relevant effects. Then, they are studied using approaches developed in [Emelianova *et al.*, 2013; Emelianova *et al.*, 2014]. We reveal possible modes of complete and partial synchronization of the oscillators, the bifurcation mechanisms in the destruction of the synchronization, and describe the parameter domains of quasi-periodic modes with different number of incommensurable frequencies. One of the main questions we discuss concerns features and distinctions of the reactive coupling in comparison with the earlier known results for dissipative coupling.

2. The Phase Model

Consider a set of equations

$$\begin{aligned}
&\ddot{x} - (\lambda - x^2)\dot{x} + x + \varepsilon(x - y) = 0,\\
&\ddot{y} - (\lambda - y^2)\dot{y} + (1 + \Delta_1)y\\
&\qquad + \varepsilon(y - x) + \varepsilon(y - z) = 0,\\
&\ddot{z} - (\lambda - z^2)\dot{z} + (1 + \Delta_2)z + \varepsilon(z - y) = 0,
\end{aligned} \tag{1}$$

where λ is a parameter controlling the intensity of the self-oscillations, Δ_1 and Δ_2 are the detuning parameters for the second and the third oscillators, and ε is the coupling constant.

If the excitation parameter λ is small enough, as well as the detuning parameters, one can apply the slow-amplitude method for the analysis of Eqs. (1). Using the standard approach [Pikovsky *et al.*, 2001; Landa, 1996], one can derive the following equations for the real amplitudes r_i and phases of the oscillators ψ_i (Landau–Stuart equations):

$$\begin{aligned}
2\dot{r}_1 &= r_1 - r_1^3 - \varepsilon r_2 \sin\theta,\\
2\dot{r}_2 &= r_2 - r_2^3 + \varepsilon r_1 \sin\theta - \varepsilon r_3 \sin\varphi,\\
2\dot{r}_3 &= r_3 - r_3^3 + \varepsilon r_2 \sin\varphi,\\
2\dot{\psi}_1 &= \varepsilon - \varepsilon\frac{r_2}{r_1}\cos\theta,\\
2\dot{\psi}_2 &= 2\varepsilon + \Delta_1 - \varepsilon\frac{r_1}{r_2}\cos\theta - \varepsilon\frac{r_3}{r_2}\cos\varphi,\\
2\dot{\psi}_3 &= \varepsilon + \Delta_2 - \varepsilon\frac{r_2}{r_3}\cos\varphi.
\end{aligned} \tag{2}$$

Here $\theta = \psi_1 - \psi_2$, $\varphi = \psi_2 - \psi_3$ are relative phases of the oscillators, and the parameters are normalized with respect to the small parameter λ [Ivanchenko *et al.*, 2004; Kuznetsov *et al.*, 2009]:

$$r \to \sqrt{\lambda} r, \quad t \to \frac{t}{\lambda}, \quad \varepsilon \to \lambda\varepsilon, \quad \Delta \to \lambda\Delta. \tag{3}$$

Subtracting pairwise the phase equations (2), we obtain the following equations for the relative phases:

$$\begin{aligned}
2\dot{\theta} &= -\varepsilon - \Delta_1 + \varepsilon\left(\frac{r_1}{r_2} - \frac{r_2}{r_1}\right)\cos\theta + \varepsilon\frac{r_3}{r_2}\cos\varphi,\\
2\dot{\varphi} &= \varepsilon + \Delta_1 - \Delta_2 + \varepsilon\left(\frac{r_2}{r_3} - \frac{r_3}{r_2}\right)\cos\varphi - \varepsilon\frac{r_1}{r_2}\cos\theta.
\end{aligned} \tag{4}$$

Let us set $r_i = 1 + \tilde{r}_i$, where the tilde designates perturbations of the stationary orbits $r = 1$. The amplitude equations are strongly damped [Pikovsky *et al.*, 2001; Rand & Holmes, 1980; Ivanchenko *et al.*, 2004; Kuznetsov *et al.*, 2009], so the orbits in a short time reach roughly the stationary amplitudes with some perturbations easily estimated from (2) as

$$\begin{aligned}
2\tilde{r}_1 &= -\varepsilon\sin\theta,\\
2\tilde{r}_2 &= \varepsilon\sin\theta - \varepsilon\sin\varphi,\\
2\tilde{r}_3 &= \varepsilon\sin\varphi.
\end{aligned} \tag{5}$$

In turn, from the phase equations (4) we obtain

$$\begin{aligned}
2\dot{\theta} &= -\varepsilon - \Delta_1 + 2\varepsilon(\tilde{r}_1 - \tilde{r}_2)\cos\theta + \varepsilon(1 + \tilde{r}_3 - \tilde{r}_2)\cos\varphi,\\
2\dot{\varphi} &= \varepsilon + \Delta_1 - \Delta_2 + 2\varepsilon(\tilde{r}_2 - \tilde{r}_3)\cos\varphi - \varepsilon(1 + \tilde{r}_1 - \tilde{r}_2)\cos\theta.
\end{aligned} \tag{6}$$

Substituting the expressions for the perturbations from (5) we get

$$\begin{aligned}
2\dot{\theta} &= -\varepsilon - \Delta_1 + \varepsilon\cos\varphi - \varepsilon^2\sin 2\theta + \varepsilon^2\left(\sin\varphi\cos\theta - \frac{1}{2}\sin\theta\cos\varphi + \frac{1}{2}\sin 2\varphi\right),\\
2\dot{\varphi} &= \varepsilon + \Delta_1 - \Delta_2 - \varepsilon\cos\theta - \varepsilon^2\sin 2\varphi + \varepsilon^2\left(\sin\theta\cos\varphi - \frac{1}{2}\sin\varphi\cos\theta + \frac{1}{2}\sin 2\theta\right).
\end{aligned} \tag{7}$$

These are the correct phase equations for three reactively coupled oscillators derived up to terms of order ε^2. Their structure is notably more complex than that for the dissipative coupling [Pikovsky *et al.*, 2001; Emelianova *et al.*, 2013]. Let us emphasize once more, that we use an assumption not only of smallness of ε, but also smallness and positiveness of λ (this is a perturbation analysis near Hopf bifurcation that will not in general be valid for large ε or λ, e.g. in the relaxation oscillation region for large λ).

In contrast to the case of two oscillators [Rand & Holmes, 1980; Ivanchenko *et al.*, 2004; Kuznetsov *et al.*, 2009], the phase equations (7) do contain terms of the first order in the coupling strength ε, however, for proper description of the synchronization effects these terms are not sufficient. Indeed, if one neglects the quadratic terms, the matrix for perturbations of the stationary state of (7) is

$$\hat{M} = \begin{pmatrix} 0, & -\varepsilon\sin\varphi \\ \varepsilon\sin\theta, & 0 \end{pmatrix}. \tag{8}$$

The trace of this matrix is zero, $S = 0$. It means that a kind of "neutral" state occurs on the border between the stable and unstable solutions (conservative dynamics). Hence, in the system there is no main resonance at all in this approximation. As follows, for the description of synchronization phenomena one has to necessarily take into account the effects of the second order in the coupling parameter.

Figure 1(a) illustrates bifurcations of equilibrium points of the system (7). In the case of

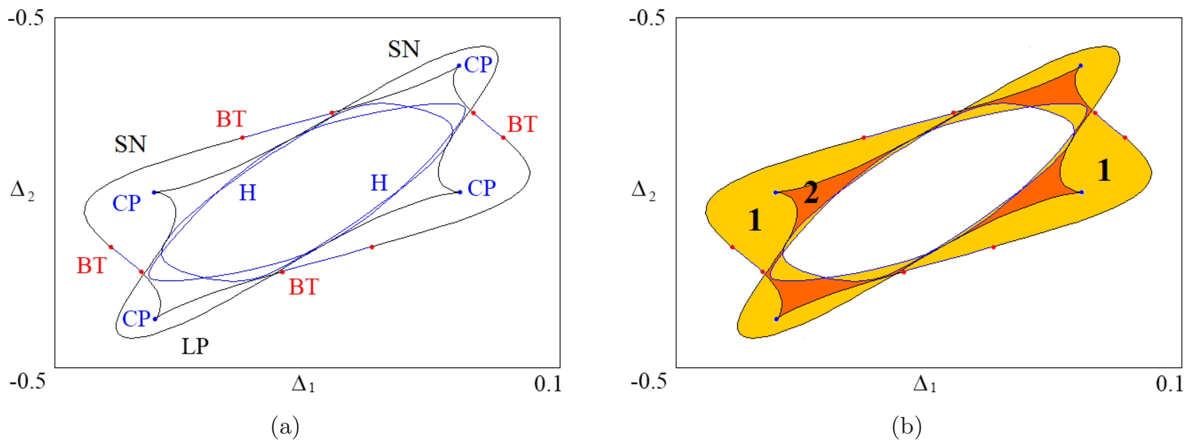

Fig. 1. (a) Bifurcation curves and points of the system (7), $\varepsilon = 0.2$ and (b) digits in fragment indicate the number of coexisting stable equilibriums.

dissipative coupling the border of the domain of complete synchronization corresponds to a curve of degenerate saddle-node bifurcations, where simultaneous merging occurs for a pair of saddles with stable and unstable nodes [Emelianova *et al.*, 2013; Anishchenko *et al.*, 2009]. For the reactive coupling the curves of the saddle-node bifurcations SN for merging a stable node and a saddle and for an unstable node and a saddle do not coincide.[1] Moreover, here a bifurcation of Andronov–Hopf is possible (designated by H), where the equilibrium point becomes unstable with the appearance of the stable limit cycle departing from it. Therefore, the region of complete synchronization in the case of reactive coupling of phase oscillators appears to be bounded both by the curves of the saddle-node bifurcations and of the line of the Andronov–Hopf bifurcations. Also we indicate in Fig. 1(a) the points of codimension two: the cusp point CP and the Bogdanov–Takens point BT. In Fig. 1(b), the domains inside which the system has one or two stable equilibria are shown using different colors. Thus, there exists the simplest multistability.

Figure 2 shows the chart of Lyapunov exponents [Emelianova *et al.*, 2013; Emelianova *et al.*, 2014] of system (7) on the parameter plane of frequency detuning of the oscillators (Δ_1, Δ_2). To draw the chart, we compute two Lyapunov exponents of the system (7), Λ_1 and Λ_2, at each pixel of the picture and attribute it with a color depending on the signature of the Lyapunov spectrum to visualize the following regimes:

(a) $\Lambda_1 < 0$, $\Lambda_2 < 0$ — the complete synchronization of three oscillators P(red),
(b) $\Lambda_1 = 0$, $\Lambda_2 < 0$ — the two-frequency quasiperiodicity T_2 (yellow),
(c) $\Lambda_1 = 0$, $\Lambda_2 = 0$ — the three-frequency quasiperiodicity T_3 (blue).

(Here the types of regimes are due to the original system (1). In this case, it is convenient to compare

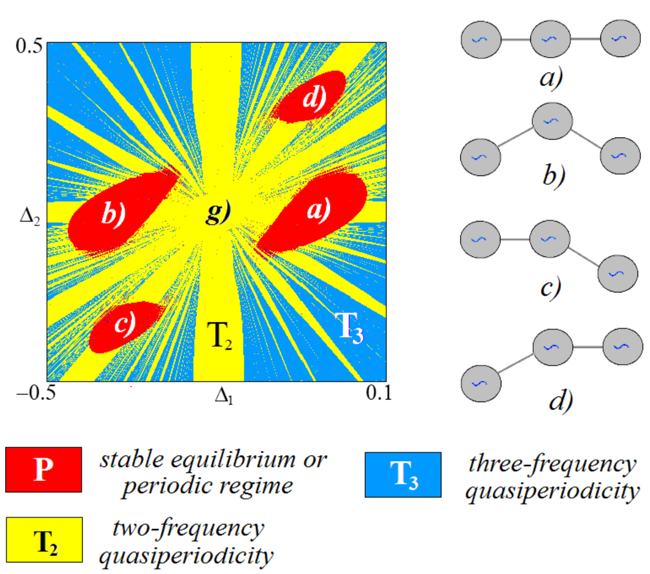

Fig. 2. Chart of Lyapunov exponents on the parameter plane and configurations of basic modes of complete synchronization of the system (7) at $\varepsilon = 0.2$. The letters designate points corresponding to the phase portraits of Fig. 3. Color palette used hereinafter for the Lyapunov charts is shown below.

[1]In Fig. 1, the curves of unstable node bifurcations are not shown to avoid cluttering the figure.

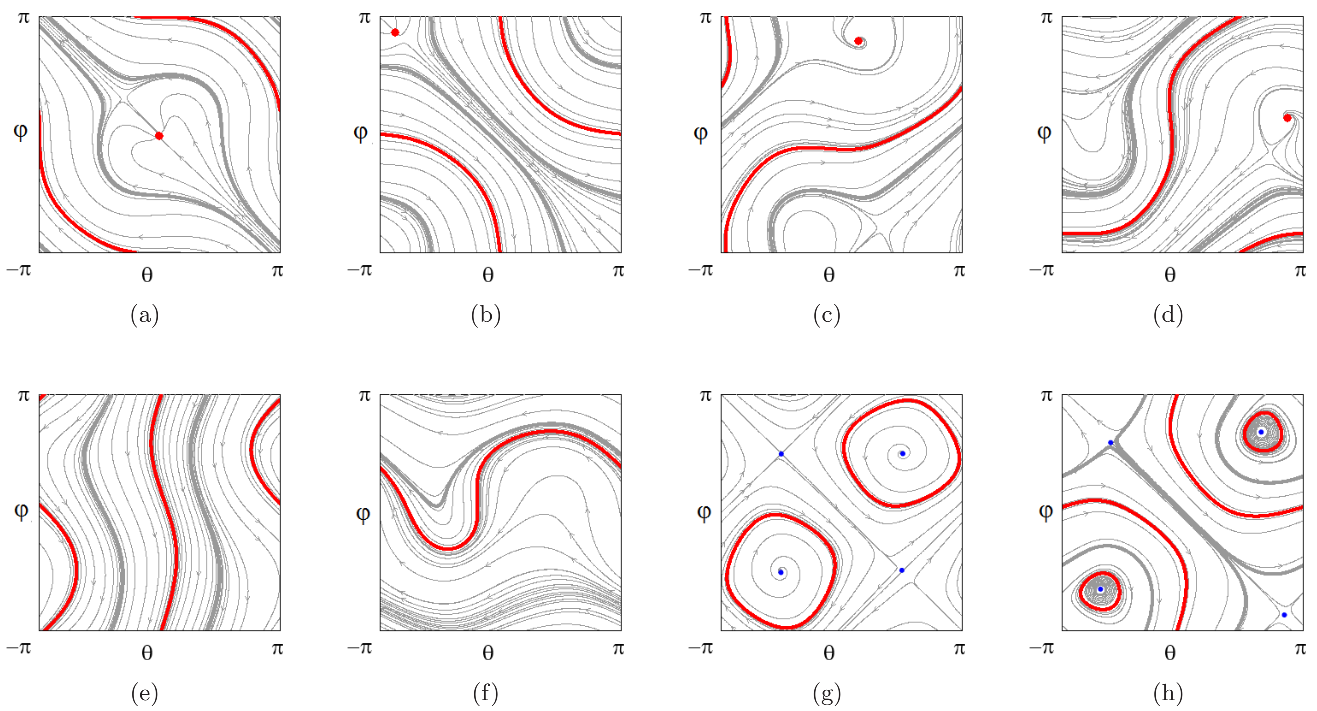

Fig. 3. Phase portraits for the system of three reactively coupled oscillators (7) at $\varepsilon = 0.2$: (a) $\Delta_1 = 0$, $\Delta_2 = 0$; (b) $\Delta_1 = -0.45$, $\Delta_2 = 0$; (c) $\Delta_1 = -0.38$, $\Delta_2 = -0.33$; (d) $\Delta_1 = 0$, $\Delta_2 = 0.36$; (e) $\Delta_1 = -0.2$, $\Delta_2 = 0.4$; (f) $\Delta_1 = 0.05$, $\Delta_2 = 0.22$; (g) $\Delta_1 = -0.2$, $\Delta_2 = 0$ and (h) $\Delta_1 = -0.3$, $\Delta_2 = 0$.

the results obtained for investigations of the phase model and the original system, see Fig. 7.)

The area of complete synchronization in Fig. 2 contains four "islands".[2] In each island we observe a specific kind of complete synchronization as illustrated in the phase portraits of Figs. 3(a) to 3(d). At the point (a) the relative phases are close to zero: $\theta \approx 0$, $\varphi \approx 0$, and this is the synchronization mode of the *in-phase* type. At the point (b) the relative phases are $\theta \approx -\pi$, $\varphi \approx \pi$; so, the first and the third oscillators are roughly in phase while the second oscillator is in the counter-phase relative to them. This is the *counter-phase* synchronization. In the remaining two islands, we observe the complete synchronization of *mixed* type. In this case, one of the pairs of the oscillators (1-2 or 2-3) are in phase, while the rest is in the counter-phase relative to them. The corresponding configurations of chain are shown in the right part of Fig. 2.[3]

Besides the region of complete synchronization, on the Lyapunov chart of Fig. 2, one can see a set of bands of two-frequency quasi-periodic regimes immersed in the domain of three-frequency regimes. Within each such band, invariant curves of different types occur in the phase plane. Say, in Fig. 3(e) the relative phase of the first and the second oscillators θ fluctuates around a certain equilibrium value, while the phase ϕ varies across the whole range of values. This is *a partial mode-locking of the first and the second oscillators*. In Fig. 3(f), one observes bounded oscillations of the relative phase of the second and third oscillators ϕ, so this is *a partial mode-locking of the second and the third oscillators*.

We can classify regions of two-frequency modes with the help of the rotation number $w = p : q$. Here p and q are numbers of intersection of the corresponding invariant curve with vertical and horizontal sides of the phase square. Only significant intersections should be used taking into consideration 2π-periodicity of phase. So for Fig. 3(e), the rotation number $w = 0 : 1$ (for both curves) and for Fig. 3(f), $w = 1 : 0$.

This classification becomes obvious if we compute a "torus map" using the numerical calculation of the factors p and q at each point in the parameter plane. The corresponding illustration is given

[2]In fact, in accordance with Fig. 1(b), the regions of complete synchronization are overlapped. On Lyapunov charts, this is not visible because of the possibility of multistability in the system.

[3]Analogous modes for coupled Bonhoeffer–van der Pol oscillators were reported in [Kryukov *et al.*, 2009].

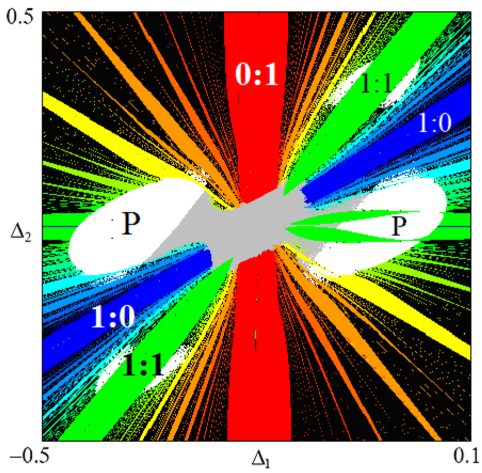

Fig. 4. "Torus map" of the two-frequency quasi-periodicity area of the system (7), $\varepsilon = 0.2$. The rule of the picture coloring is explained in text.

in Fig. 4. We use the following rule of coloring. Blue color is associated with regime of the rotation number $w = 1 : 0$. Decreasing of the rotation number corresponds to a piecemeal transformation of this color to green mode $w = 1 : 1$. Then green color is gradually transformed into red for the mode with the rotation number $w = 0 : 1$. Stable equilibriums are shown in white and the other modes — in black. Light gray color corresponds to the regime with contractible limit cycles when the invariance curve has no significant intersections with the sides of the phase of a square.

The origin of the two frequency bands with a common symmetry center in Fig. 2 may be explained by the presence of various possible resonances in the system. To substantiate this point, outline *the partial frequencies* ω_i of the system (6). If we "switch off" in each equation all other oscillators, in the linear approximation we obtain from (6)

$$\omega_1 = \frac{\varepsilon}{2}, \quad \omega_2 = \varepsilon + \frac{\Delta_1}{2}, \quad \omega_3 = \frac{\varepsilon}{2} + \frac{\Delta_2}{2}. \qquad (9)$$

A feature of the reactive coupling is that it shifts the partial frequencies by a value of order ε (the shift for the central oscillator is twice as large as for the other ones because it interacts with two rather than one neighbor).

Now, let us write out the main resonances for the partial frequencies. In round brackets, the respective conditions are indicated in terms of the frequency detuning parameters:

$$\begin{aligned} &\omega_1 = \omega_2(\Delta_1 = -\varepsilon), \\ &\omega_2 = \omega_3(\Delta_2 = \Delta_1 + \varepsilon), \\ &\omega_1 = \omega_3(\Delta_2 = 0), \\ &\omega_1 + \omega_3 = 2\omega_2(\Delta_2 = 2\Delta_1 + 2\varepsilon). \end{aligned} \qquad (10)$$

The resonance conditions (10) determine the centers for the wide bands of two-frequency modes in Fig. 2. Additional resonances are possible too, say, $\omega_1 + \omega_2 = 2\omega_3$, etc. They correspond to more narrow bands in Fig. 2. The resonances of higher order give rise to invariant curves with larger number of intersections with the sides of the phase square.

Note that the resonances $\omega_1 = \omega_3$ and $\omega_1 + \omega_3 = 2\omega_2$ determine lines of symmetry on the parameter plane, see Figs. 1 and 2. This is due to symmetry of these resonance conditions with respect to permutation of the first and third oscillators. Say, under the condition $\omega_1 = \omega_3$, i.e. $\Delta_2 = 0$, Eqs. (7) are transformed from one to the other under the variable change $\theta \leftrightarrow -\varphi$. As a result, the phase portraits are symmetric about the line $\varphi = -\theta$, see Figs. 3(g) and 3(h). Analogously, with the condition $\Delta_2 = 2(\Delta_1 + \varepsilon)$ the equations are invariant with respect to the variable change $\theta \leftrightarrow \varphi + \pi$.

An interesting case is the equality of all the partial frequencies $\omega_1 = \omega_2 = \omega_3$ that correspond to the symmetry center on the parameter plane $\Delta_1 = -\varepsilon$, $\Delta_2 = 0$. The phase portrait at this point is shown in Fig. 3(g), and at the point close to it in Fig. 3(h). In this case, invariant curves of other kind arise, which are distinct in their topological properties. The curves in Fig. 3(g) may be called *contractible*, while those in Figs. 3(a)–3(f) are called *rotational* [Baesens *et al.*, 1991] (for the dissipative coupling, the second case is typical [Emelianova *et al.*, 2013]). In the first case, we have a limit cycle going around the unstable equilibrium point. Then, both the relative phases fluctuate about some mean value. This mode may be characterized as *a partial mode-locking of all three oscillators*. The frequency spectrum produced by the system (1) will contain not only the basic frequency, but also a set of components associated with an additional new time scale, the period of travel of the representative point around the limit cycle of the phase model.

Contractible limit cycles can occur, as we have already noted, as a result of Andronov–Hopf bifurcation. In addition, the system can demonstrate nonlocal bifurcations when the resulting limit cycle arises from the separatrix loop of the saddle.

Note that several types of multistability are possible in the system with reactive coupling as seen from Fig. 3. Figures 3(a)–3(d) correspond to the situation when an equilibrium state in the phase space corresponding to the complete synchronization coexists with an invariant curve corresponding to a two-frequency quasi-periodic regime.[4] In Fig. 3(e) two regimes coexist (the in-phase mode and the counter-phase mode) corresponding to partial mode-locking of the first and second oscillators. In turn, in Fig. 3(g) there coexist two regimes of partial synchronization of all three oscillators.

Multistability affects the appearance of the Lyapunov chart. Namely, when choosing different initial conditions one can observe either periodic or quasi-periodic regimes, as can be seen from a comparison of Figs. 1 and 2.

3. Dynamics of the Original System

Let us illustrate the effectiveness of the phase model. For this purpose, we represent the results of the bifurcation analysis of the original system (1) at $\lambda = 0.1$, Fig. 5. In accordance with the renormalization rules (3) the coupling parameter $\varepsilon = 0.02$ is selected, which allows to compare Fig. 5 for the original system and Fig. 1 for the phase model. Now instead of a saddle-node bifurcation of equilibria there is a corresponding bifurcation of limit cycles SNC, instead of the Andronov–Hopf bifurcation–Neimark–Sacker bifurcation NS, and instead of Bogdanov–Takens points–points of 1:1 resonance R1. From Figs. 1 and 5, we can see that the figures are similar to each other, which indicates the effectiveness of the phase model for these values of coupling parameter.

It should be noted that the effectiveness of the phase model will increase with a decrease in the coupling parameter ε. With increase of coupling parameter its efficiency falls. Figure 6 illustrates this fact, demonstrating bifurcation lines of the phase model and the original system (1). For the phase model we select coupling parameter $\varepsilon = 0.6$, so for the original system we have $\varepsilon = 0.06$ [taking into account $\lambda = 0.1$ rescaled on (3)]. Some common features — the presence of four lobes — remain. However, the pictures are different in details. Namely, external boundaries of lobes are now mainly the Neimark–Sacker lines NS. Lines of saddle-node bifurcations of limit cycles SNC form only small segments of the boundary of complete synchronization area in the vicinity of the cusp points CP associated with bistability areas. One more new feature is the appearance of double Neimark–Sacker bifurcation points NS–NS.

Chart of Lyapunov exponents of the original system (1) is presented in Fig. 7 for $\lambda = 0.1$, $\varepsilon = 0.06$. An enlarged fragment of this chart in Fig. 7(b) should be compared with Fig. 6(b).

With the growth of the λ control parameter both the Landau–Stuart and phase models will get from bad to worse. In Fig. 8(a) the Lyapunov chart is shown for the system (1) for the case $\lambda = 1$, $\varepsilon = 0.6$. Now the picture is much more complex. Figure 8(b) shows a new effect: at the intersection of numerous bands of two-frequency modes there

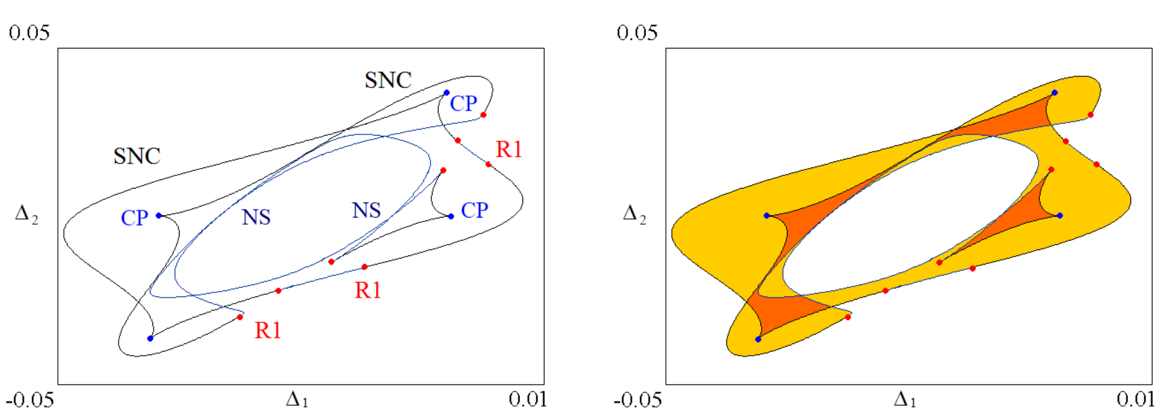

Fig. 5. Bifurcation curves and points of the system (1), $\lambda = 0.1$, $\varepsilon = 0.02$.

[4]Actually, Fig. 2 is a composition of four charts associated with four variants of the initial conditions and four possible modes of complete synchronization in the system.

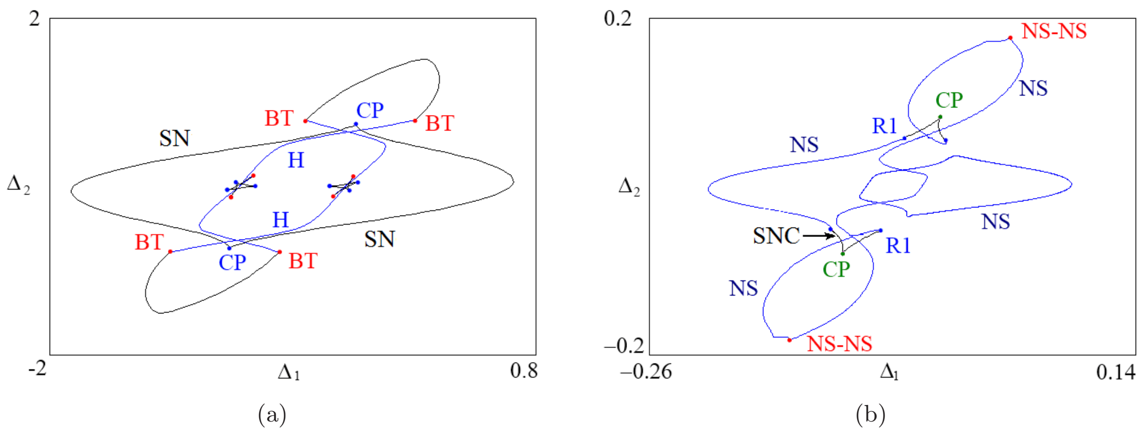

Fig. 6. (a) Bifurcation lines and dots of phase model (7), $\varepsilon = 0.6$ and (b) similar illustration for the original system (1), $\lambda = 0.1$, $\varepsilon = 0.06$.

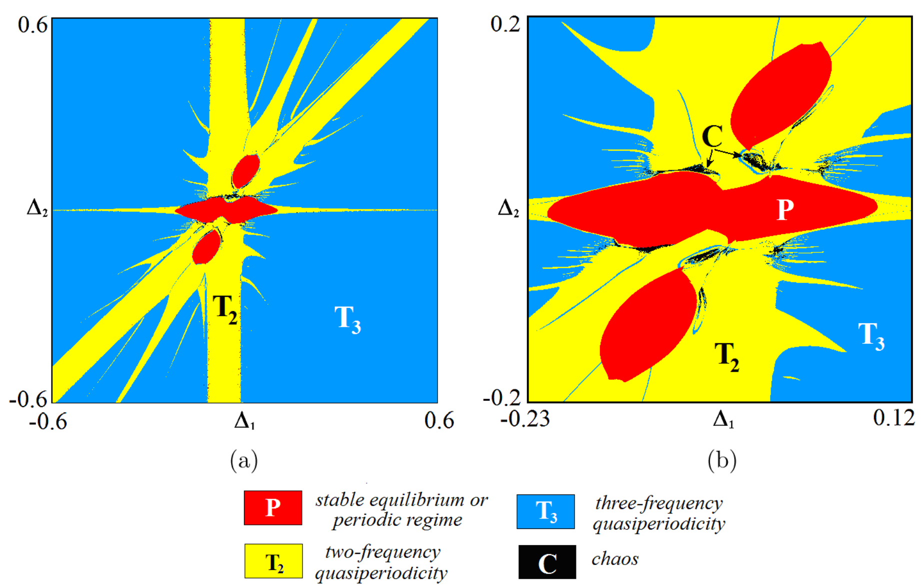

Fig. 7. The chart of Lyapunov exponents and its magnified fragment of the system (1), $\lambda = 0.1$, $\varepsilon = 0.06$. Presented below, the color layout is also used for Fig. 8.

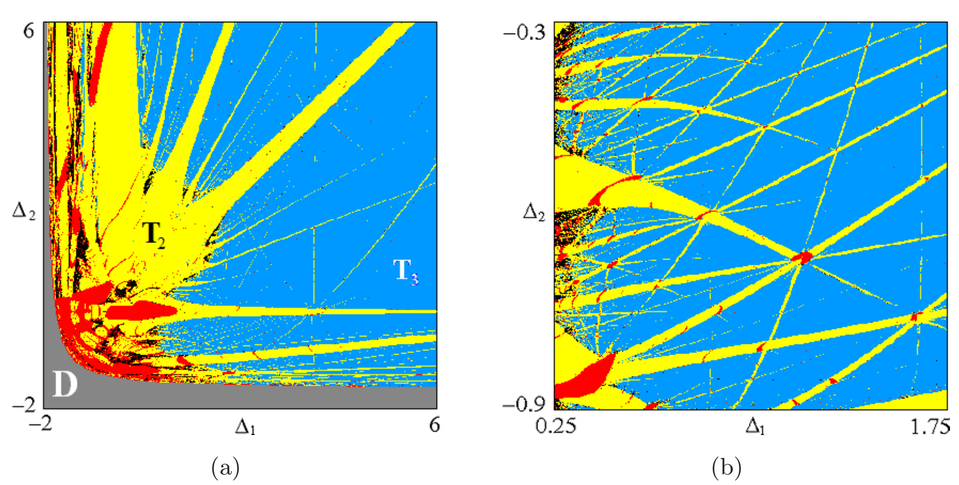

Fig. 8. The chart of Lyapunov exponents and its magnified fragment of the system (1), $\lambda = 1$, $\varepsilon = 0.6$.

are located regions of higher resonances, to which there correspond periodic regimes. This is a characteristic *resonance Arnold web* [Broer *et al.*, 2008]. Letter D denotes the area of the trajectories' escape to infinity.

4. Conclusion

The results presented here may be of interest for systems such as ion traps [Lee & Cross, 2011], biophysical systems [Rompala *et al.*, 2007], etc. At the same time, due to universality of our models, they are important as well in the general theory of synchronization. The main result is that in the description of reactive coupling in the framework of the slow amplitudes approach, it is of principal importance to account for the effects of the second order in the coupling constant. The region of complete synchronization consists of four "islands" in which the synchronization modes are observed corresponding to in-phase, counter-phase, and mixed types of oscillations in the chain. The bifurcation picture of the model formulated in terms of phase variables is significantly different from the case of dissipative coupling. In particular, Andronov–Hopf bifurcation is possibly responsible for the occurrence of partial synchronization regimes of all three oscillators. For the arrangement of the parameter plane of the frequency detunings, resonances between the partial frequencies are important, and for the reactive coupling the resonance conditions appear to depend on the coupling strength. One more feature is that the quasi-periodic modes of various types can coexist with complete synchronization, giving rise to many kinds of multistability (coexistence of different attractors).

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