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The Simplest Map with Three-Frequency Quasi-Periodicity and Quasi-Periodic Bifurcations

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We propose a new three-dimensional map that demonstrates the two- and three-frequency quasi-periodicity. For this map, all basic quasi-periodic bifurcations are possible. The study was realized by using Lyapunov charts completed by plots of Lyapunov exponents, phase portraits and bifurcation trees illustrating the quasi-periodic bifurcations. The features of the three-parameter structure associated with quasi-periodic Hopf bifurcation are discussed. The comparison with nonautonomous model is carried out.

Keywords: Quasi-periodic oscillations; quasi-periodic bifurcations; torus map.

1. Introduction

Quasi-periodic oscillations are widespread in nature and technology [Blekhman, 1988; Landa, 1996; Pikovsky *et al.*, 2001; Anishchenko *et al.*, 2007a]. Examples of quasi-periodic behavior are found in electronics [Glazier & Libchaber, 1988], neurodynamics [Izhikevich, 2000], astrophysics [Lamb *et al.*, 1985], physics of lasers [Mel'nikov *et al.*, 1991], geology [Didenko, 2011] and other areas of science.

In the simplest case, the quasi-periodic oscillations are characterized by the presence of two incommensurable frequencies. In the phase space, attractors corresponding to such oscillations have the form of invariant tori [Anishchenko, 1995; Shil'nikov *et al.*, 2001; Broer & Takens, 2010]. There are more complicated cases of greater number of incommensurable frequencies — then multifrequency oscillations are observed which correspond to invariant tori of higher dimensions. Invariant tori may undergo a variety of bifurcations. In this case we speak about quasi-periodic bifurcations [Broer *et al.*, 2008a, 2008b; Vitolo *et al.*, 2011]. The basic ones are the next three bifurcations:

- the saddle-node bifurcation. Collision of stable and unstable tori leads to the abrupt birth of higher-dimensional torus.
- the quasi-periodic Hopf bifurcation. Torus of higher dimension is born softly.
- the torus-doubling bifurcation.

Currently there are no reliable identification algorithms for such bifurcations and their search is executed by means of the characteristic dependence of Lyapunov exponents on parameter [Broer *et al.*, 2008a, 2008b; Vitolo *et al.*, 2011].

While examining the quasi-periodic bifurcations the selection of the model for studying plays a crucial role. Some aspects can be studied based on nonautonomous systems. However, such systems form a separate, special class. As concerning the known autonomous models, the number of low-dimensional variants is not very large. Several examples of suitable generators have been proposed recently. The first example, obviously, is a Chua's circuit, which is described by specific piecewise-linear characteristics [Matsumoto *et al.*, 1987]. In [Nishiuchi *et al.*, 2006] a generator based on the modified Bonhoeffer–van der Pol system is offered. Also we know a modification of climate Lorenz model — the Lorenz-84 low-order atmospheric circulation model [Shil'nikov *et al.*, 1995]. Another radiophysical example implemented as a four-dimensional system is a modified Anishchenko–Astakhov generator [Anishchenko & Nikolaev, 2005; Anishchenko *et al.*, 2006, 2007a]. In [Kuznetsov *et al.*, 2010; Kuznetsov *et al.*, 2013, 2015; Kuznetsov & Stankevich, 2015], a family of simple generators of quasi-periodic oscillations described by three-dimensional models was suggested and studied (including experiment). There have been studied cases of autonomous dynamics as well as on the generator under external forcing and dynamics of coupled oscillators [Anishchenko *et al.*, 2006, 2007b; Kuznetsov *et al.*, 2013; Stankevich *et al.*, 2015].

It is known that flow systems (differential equations) are quite difficult for research, especially when concerning the study of delicate effects like quasi-periodic bifurcations. Therefore, it is natural to deal with a simpler model such as discrete maps. In this case, the image of two-frequency oscillations in the phase space is an invariant curve. As for flows, the same object emerges in the Poincaré section of torus. (Thus frequently such invariant curves are also referred to as tori.) The higher is the map dimension the higher may be dimension of the torus. Most recently, appropriate investigations of model maps have been undertaken. In paper [Kuznetsov *et al.*, 2012b] the four-dimensional system of two coupled universal maps with Neimark–Sacker bifurcation is examined. In [Sekikawa *et al.*, 2014; Kamiyama *et al.*, 2014], a four-dimensional map representing two logistic maps with delay was studied. The authors of the papers [Hidaka *et al.*, 2015a; Hidaka *et al.*, 2015b] undertook an analogous study of six-dimensional system in the form of three logistic maps with delay. In [Adilova *et al.*, 2013] the six-dimensional model which represents two coupled discrete versions of the Rössler system is examined.

However, the dimension of the above-mentioned systems is too large compared to the dimension which is formally necessary to observe two- and three-frequency quasi-periodicity. An exception is the paper [Dementyeva *et al.*, 2014] that studied the three-dimensional system in the form of three coupled in a ring of logistic maps with a specific coupling. Nevertheless, the analysis of the problem in this paper seems inadequate. The noteworthy model from [Broer *et al.*, 2008a, 2008b; Vitolo *et al.*, 2011] was used for studying quasi-periodic bifurcations, but its construction is a complex formal procedure. Although the model describes the very important points, the discussed picture is still not quite complete. (The authors carried out only two-parameter analysis, they mainly focused on the description of resonance 1:5.)

It can be noted that in [Anishchenko *et al.*, 1994] the coupled logistic maps with quasi-periodic forcing are studied. The research concerns bifurcations and mechanisms of transition to chaos through the destruction of three-dimensional torus. However, not all the essential points have been investigated in detail. Moreover, this model belongs to the class of nonautonomous systems, i.e. systems with external forcing.

Thus, there is a problem to construct autonomous models in the form of maps with the minimum necessary dimension equal to three that allows to study the properties of quasi-periodic dynamics and quasi-periodic bifurcations (we use the world "simplest" in the paper title to emphasize the lowest dimension of our autonomous model). In the present paper, such a model is represented by a discrete analogue of quasi-periodic generator [Kuznetsov *et al.*, 2013, 2015]. We focus on the research of the structure of the parameter space focusing on the phenomena associated with quasi-periodic Hopf bifurcation.

2. Torus Map Construction

Let us consider the generation of quasi-periodic oscillations [Kuznetsov *et al.*, 2013, 2015]:

$$\begin{aligned} &\ddot{x} - (\lambda + z + x^2 - \beta x^4)\dot{x} + \omega_0^2 x = 0, \\ &\dot{z} = b(\varepsilon - z) - k\dot{x}^2. \end{aligned} \tag{1}$$

The multiplier ahead of the derivative $\dot{x}$ contains the parameter λ which characterizes the depth of the positive feedback in the oscillator, the nonlinear term x^2 stimulates the excitation of oscillations and the term x^4 responsible for the saturation of the oscillations at large amplitudes. The dynamics of the variable z can occur linearly at a speed b or undergo the nonlinear saturation due to the term $k\dot{x}^2$.

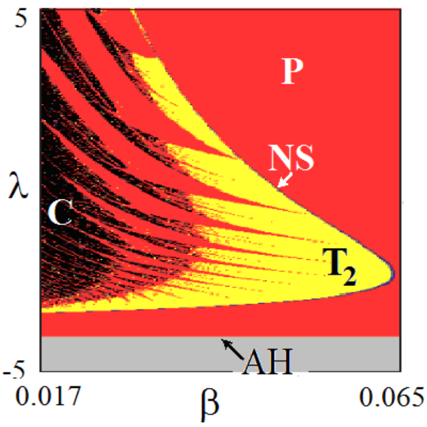


Fig. 1. Lyapunov chart for generating of quasi-periodic oscillations (1), $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\omega_0 = 2\pi$.

The model (1) has an equilibrium $x_0 = y_0 = 0$, $z_0 = \varepsilon$. It is easily verified that this equilibrium undergoes Andronov–Hopf bifurcation AH at $\lambda = -\varepsilon$ leading to the birth of a limit cycle. By increasing the parameter λ, the Neimark–Sacker bifurcation NS consisting in the birth of a two-dimensional torus is possible. In Fig. 1, we demonstrate the Lyapunov chart for system (1) where different colors indicate the areas of periodic regimes P, the two-frequency quasi-periodicity T_2 and chaos C. We can see the curve of Neimark–Sacker bifurcation NS with the adjoined set of Arnold tongues immersed in the region of quasi-periodic oscillations.

Let us rewrite Eqs. (1) in the form of first-order system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= (\lambda + z + x^2 - \beta x^4)y - \omega_0^2 x,\\ \dot{z} &= b(\varepsilon - z) - ky^2\end{aligned} \tag{2}$$

and construct a discrete analog of Eqs. (2). For this purpose, we use a substitution for time derivatives by finite differences similarly to [Zaslavskii *et al.*, 1988; Zaslavsky, 2007; Arrowsmith *et al.*, 1993]. The transition to finite differences will provide some additional characteristic time scale — a discretization step, which usually leads to new types of dynamics. The result is a model that we call a torus map:

$$\begin{aligned}x_{n+1} &= x_n + h \cdot y_{n+1},\\ y_{n+1} &= y_n + h \cdot ((\lambda + z_n + x_n^2 - \beta x_n^4)y_n - \omega_0^2 x_n),\\ z_{n+1} &= z_n + h \cdot (b(\varepsilon - z_n) - ky_n^2).\end{aligned} \tag{3}$$

Here h is a discrete time step. Note that for the first equation we use a semi-explicit Euler scheme, i.e. we take the value of the variable y in $(n+1)$th moment. This discretization usually leads to more physically correct models [Morozov, 2005].

3. Properties of Torus Map with Quasi-Periodic Dynamics

Let us study the obtained map. We use the same set of parameters as for Fig. 1 and will gradually increase the discretization parameter h. In the center of Fig. 2 a numerically calculated Lyapunov chart for system (3) at the value $h = 0.05$ is shown. Different colors in the chart denote the following regions defined in accordance with the spectrum of Lyapunov exponents $\Lambda_1, \Lambda_2, \Lambda_3$:

(a) P — periodic regimes (cycles), $\Lambda_1 < 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$;
(b) T_2 — two-frequency quasi-periodicity, $\Lambda_1 = 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$;
(c) T_3 — three-frequency quasi-periodicity, $\Lambda_1 = 0$, $\Lambda_2 = 0$, $\Lambda_3 < 0$;
(d) C — chaotic regimes, $\Lambda_1 > 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$;
(e) HC — hyperchaotic regimes, $\Lambda_1 > \Lambda_2 > 0$, $\Lambda_3 < 0$;
(f) D — a divergence of trajectories.

Due to the smallness of the discretization parameter h the structure of the chart is partly qualitatively similar to that of the original flow system (1). (The smaller h the better an approximation to the flow system.) However the discretization leads to the replacement of periodic regimes of flow system (1) by two-frequency regimes and two-frequency by three-frequency ones in Fig. 1. Thus in Fig. 2 can see a picture of tongues of two-frequency regimes that in configuration are similar to traditional Arnold tongues. The mentioned set of tongues immerses in a region of three-frequency tori. The tongues in Fig. 2 correspond to resonant two-frequency tori lying on the surface of three-frequency torus.

In Fig. 2, we show examples of phase portraits at the various points of the parameter plane. Below right to the line QH torus looks like a simple oval. In the three-frequency region this oval is smeared. Inside the tongues of two-frequency regimes the attractors have the form of closed invariant curves. In different tongues on the plain (x, y) these curves differ in the number of turns around the origin. Such curves replace the simple limit cycles in the

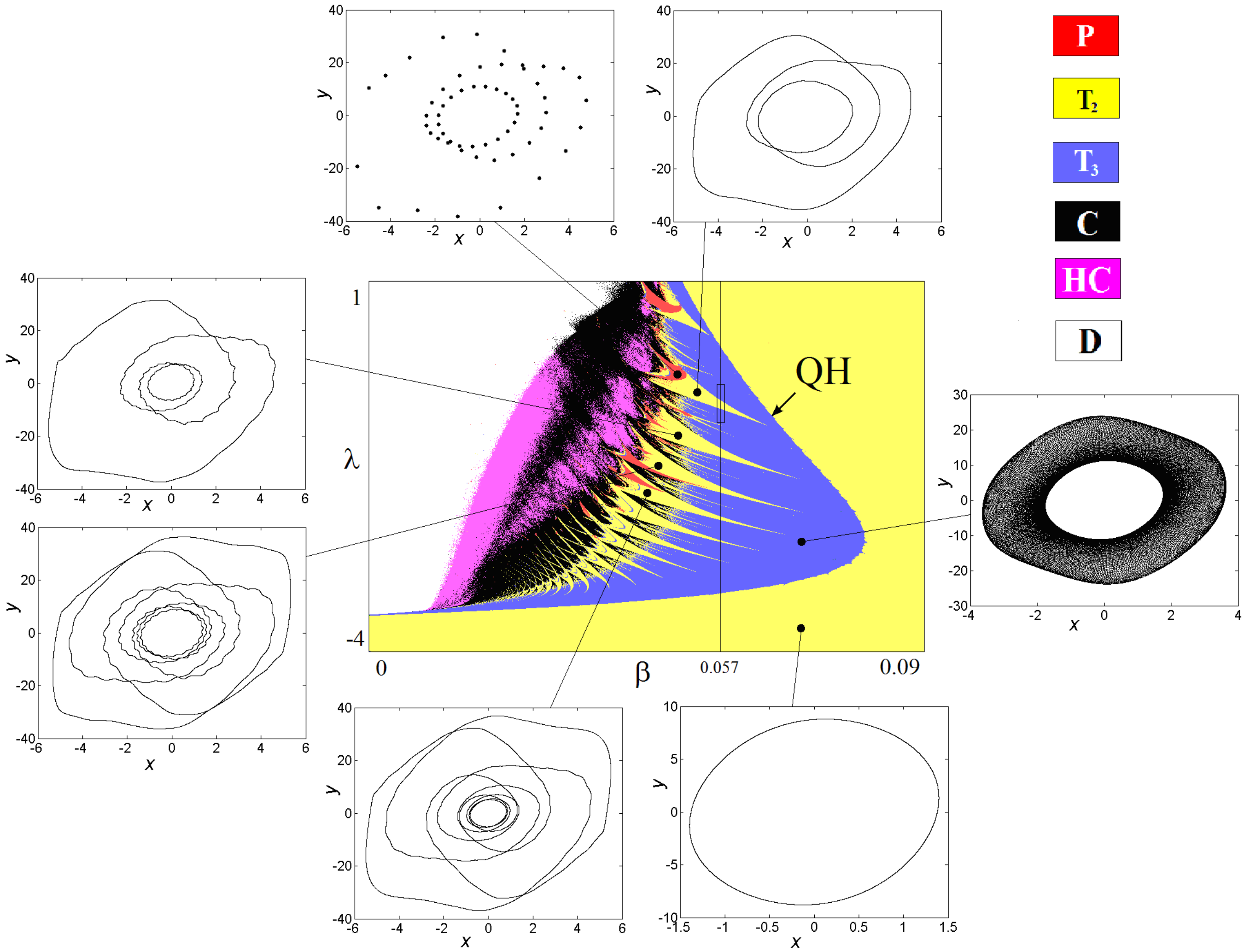


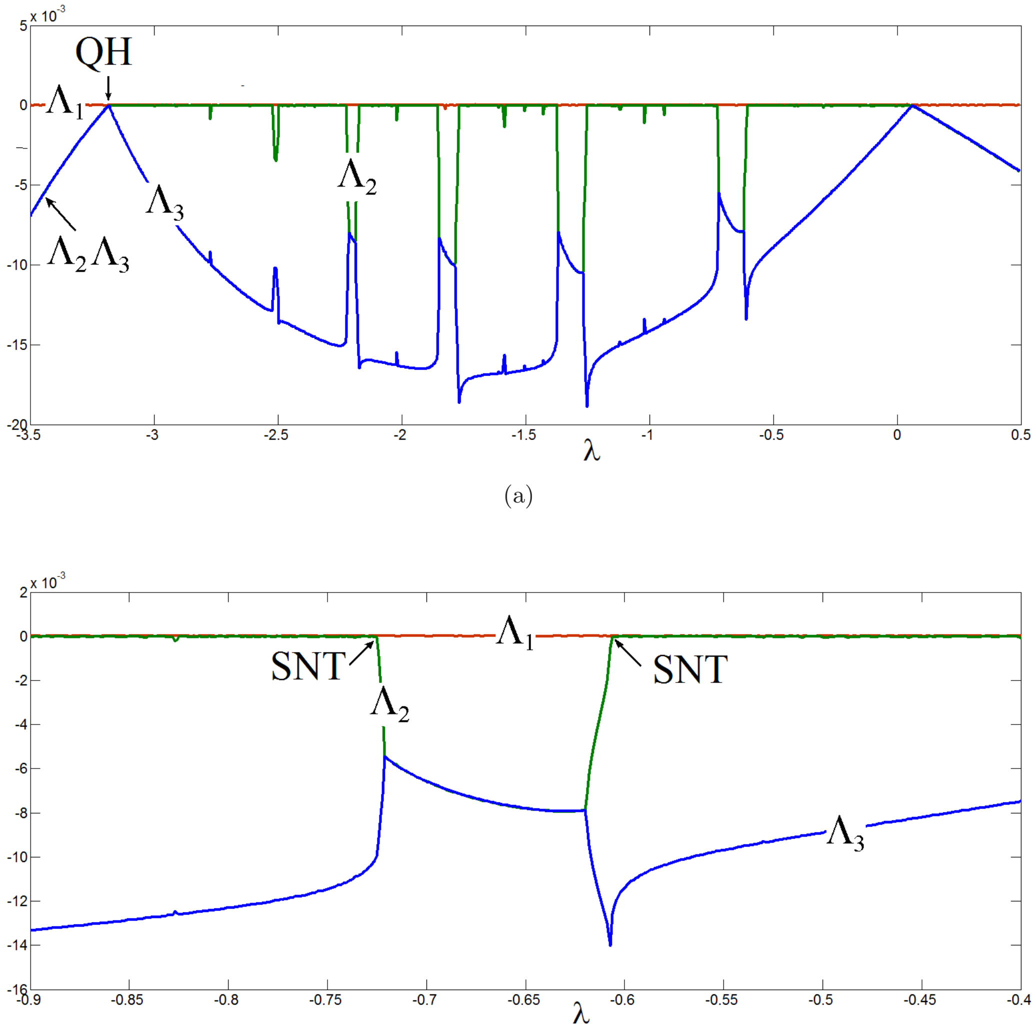
Fig. 2. Lyapunov chart of the torus map (3) and typical phase portraits. QH is a line of quasi-periodic Hopf bifurcation. Discretization parameter value $h = 0.05$.

Poincaré section of the original model (1) and their shape indicates that discretization even with a small step h leads to a specific modification of the observed structure.

In turn inside tongues of two-frequency tori the areas of periodic regimes arise. In this case on the invariant curve of complex shape there is a set of points of appropriate long-period cycle, i.e. such regimes are resonant with respect to the corresponding two-frequency torus.

Let us now discuss bifurcations of quasi-periodic regimes. With this goal we turn to Lyapunov exponent plots in Fig. 3 calculated along those selected in Fig. 2, line $\beta = 0.057$. Such a line crosses the three-frequency periodicity region from the bottom to the top. At the point QH a two-frequency torus ($\Lambda_1 = 0$) undergoes bifurcation. As can be seen from Fig. 3, the feature of this bifurcation is that below the threshold the exponents Λ_2 and Λ_3 are equal, $\Lambda_2 = \Lambda_3$. At the bifurcation point, both these exponents vanish. Beyond the bifurcation point, exponents do not coincide: the second exponent is equal to zero, $\Lambda_2 = 0$, and the third one becomes negative, $\Lambda_3 < 0$. Exactly at the point of bifurcation the condition $\Lambda_1 = \Lambda_2 = 0$ is fulfilled and a three-frequency torus emerges. This bifurcation is called quasi-periodic Hopf bifurcation QH. Its distinguishing feature is the realization of criterion of coincidence of two exponents beyond the bifurcation point [Vitolo *et al.*, 2011; Broer *et al.*, 2008a].

With increasing of control parameter λ the route $\beta =$ const on the chart in Fig. 1 crosses many tongues of two-frequency tori. In the plot of Fig. 3(a) such tongues manifest themselves as dips in the graphs of the second exponent. The



(a)

(b)

Fig. 3. Three Lyapunov exponents of the model (3) versus (a) parameter λ and (b) the magnified fragment. Also seen is the location of quasi-periodic Hopf bifurcation point QH and saddle-node torus bifurcations SNT. Discretization parameter $h = 0.05$, parameter $\beta = 0.057$.

boundaries of these areas are formed by lines of saddle-node torus bifurcation SNT. One of the deepenings in the enlarged view is demonstrated in Fig. 3(b). Distinct characteristics of SNT bifurcation is that the second Lyapunov exponent Λ_2 vanishes, but values Λ_2, Λ_3 are not equal to each other [Vitolo *et al.*, 2011; Broer *et al.*, 2008a]. Herewith the third exponent Λ_3 remains always negative [see Fig. 3(b)]. On the other side of the tongue, such a bifurcation takes place in reverse order.

Accordingly we can indicate the curve of quasi-periodic Hopf bifurcation QH in Fig. 2 separating three-frequency and two-frequency regions. In the discrete model (3) such a line replaces the line of Neimark–Sacker bifurcation NS in the flow-prototype (1) displayed in Fig. 1.

It should be noted that quasi-periodic Hopf bifurcation is essentially three-parameter phenomenon in contrast to the traditional two-parameter Neimark–Sacker bifurcation. The physical nature of this fact is explained by adding a parameter associated with additional frequency. In model (3) such a parameter can be a discretization step h, which is responsible for an additional time scale. Therefore, we will increase the parameter h and trace the emerging changes in the structures of domains on the examined parameter plane.

Let us discuss the case $h = 0.1$, Fig. 4. There are significant qualitative changes on the parameter plane. A distinctive set of Arnold-type tongues disappears. From the line of a quasi-periodic Hopf bifurcation QH the bands of two-frequency regimes

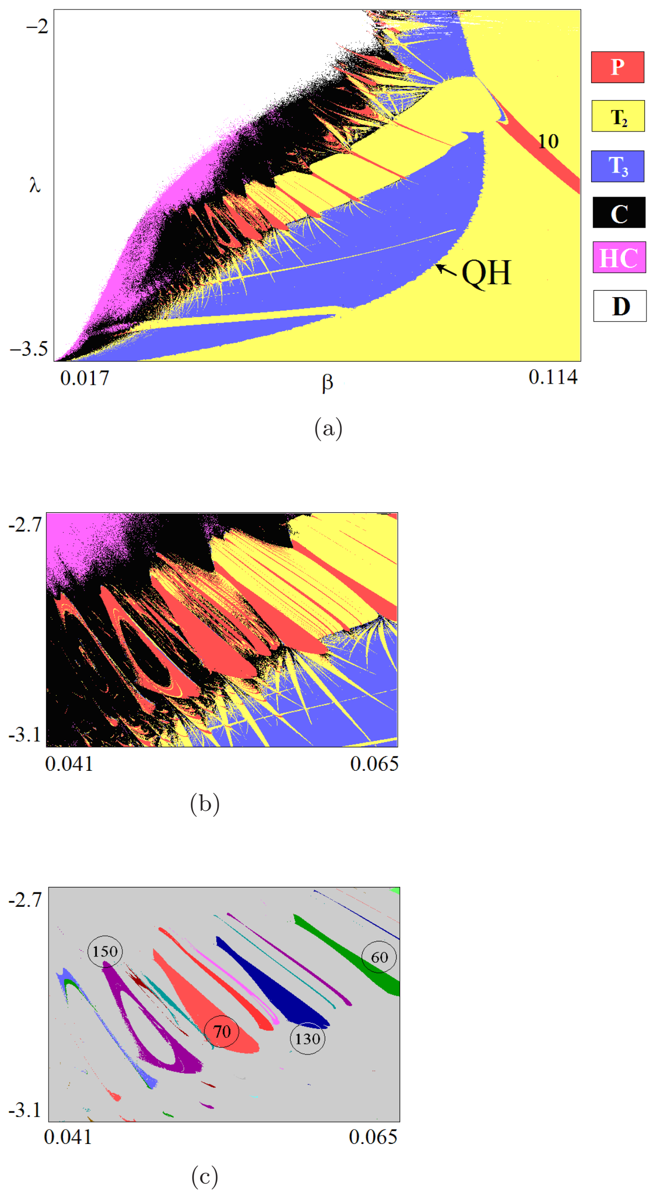


Fig. 4. (a) Lyapunov chart of torus map (3) for parameter of discretization $h = 0.1$, (b) its enlarged segment and (c) the chart of regimes of torus map.

are issued. The crosswise bands of periodic regions (exact resonances) are built in these two-frequency areas. In the enlarged part of Lyapunov chart, Fig. 4(b), we see that the mentioned resonances generate secondary sets of the fan-shaped two-frequency tongues, immersed in a three-frequency region.

To better visualize and distinguish periodic regimes we mark by different colors and numbers the cycles of various periods [Fig. 4(c)]. Gray color corresponds to all the nonperiodic regimes. It can be seen that the built-in areas of periodic regimes have different periods, and their values are large enough.

Another representative fact in Fig. 4 consists in period-10 tongue approaching the mentioned two-frequency band from the main two-frequency area as described in [Broer *et al.*, 2008b].

In Fig. 5, we demonstrate the examples of phase portraits. There the cycle of period-10 can be seen and also its transformations within the corresponding two-frequency resonance region. An emergence of small isolated ovals may be observed around the elements of the period-10 cycle. Cycles of very high periods inside narrow regions of periodic regimes are very typical. Thus, the structure of the parameter plane is different from that in Fig. 2.

Plots of Lyapunov exponents calculated along the vertical line in Fig. 5 are shown in Fig. 6. In this case, we observe a quasi-periodic Hopf bifurcation QH. Another illustration of such bifurcation is a bifurcation tree presented in Fig. 6(b) of appropriate scale. One can note the "smeary" crown of the tree that signalized about the quasi-periodic dynamics. At the point of quasi-periodic Hopf bifurcation QH, we observe the widening of the bifurcation tree, and it occurs in a gentle way.

Figure 7 presents the enlarged fragments of Fig. 6. We can see a strong irregularity of diagrams due to the increasing complexity of the parameter plane. There are many alternating regions of two-frequency quasi-periodic and periodic regimes. Nevertheless, there are distinguished areas of resonance tori bounded by saddle-node bifurcation points SNT.

In Fig. 7(b), we show the part of the bifurcation tree to conclude that in contrast to the points QH, at saddle-node torus bifurcation SNT points, the expansion of the crown occurs abruptly. It happens due to the nature of the bifurcation — stable and saddle two-frequency tori collide and three-frequency torus occurs abruptly [Vitolo *et al.*, 2011; Broer *et al.*, 2008a]. On the bifurcation tree, also clearly seen is the resonance window of the periodic regime.

The chart of Lyapunov exponents for greater value of parameter h is shown in Fig. 8. Now the band of period-10 is not visible. The bottom of the largest two-frequency tongue is limited by the bifurcation line, where the corresponding Lyapunov exponent vanishes. On the chart this line appears as a thin blue strip. Its type can be determined by studying the phase portraits: this

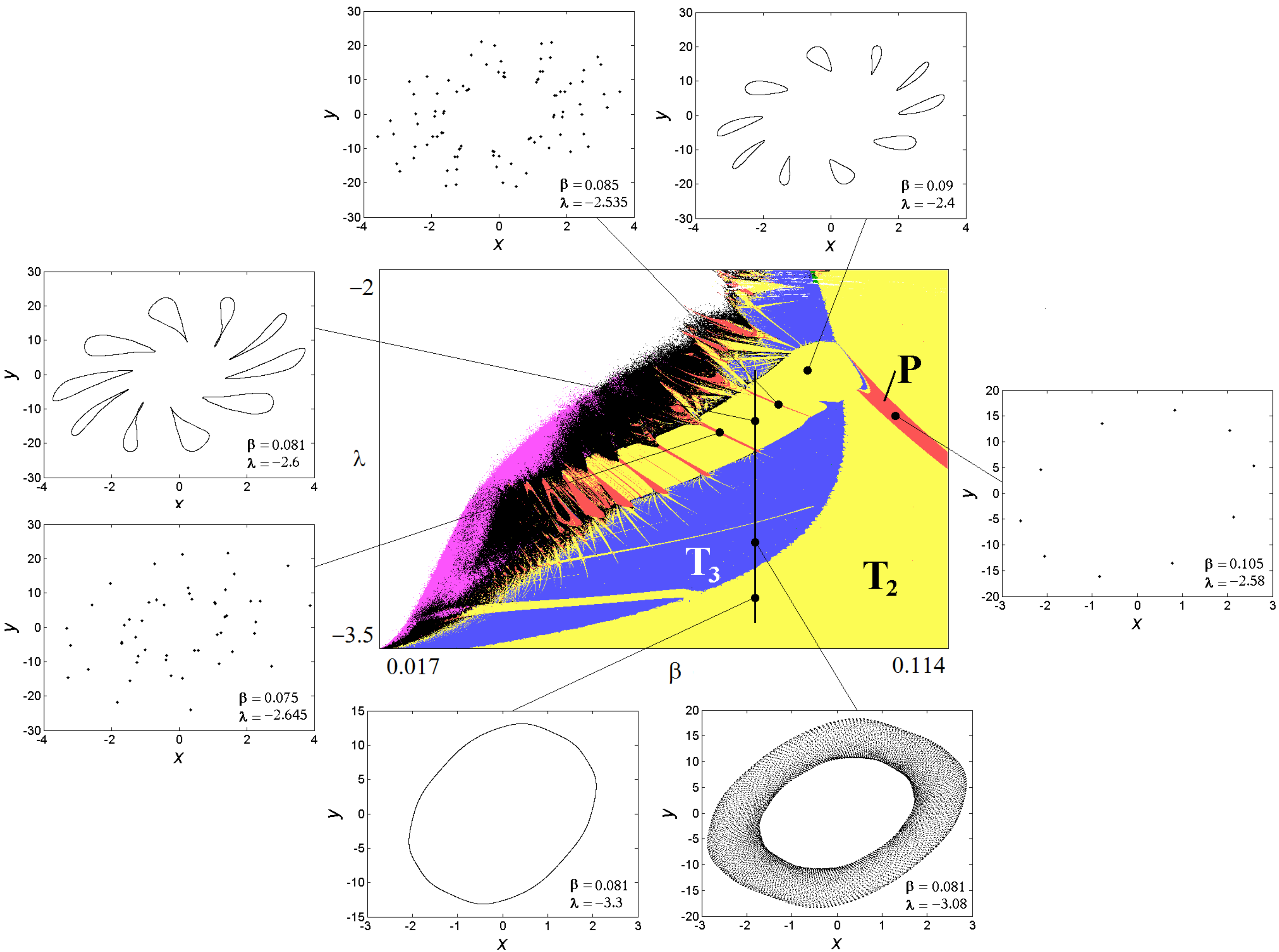
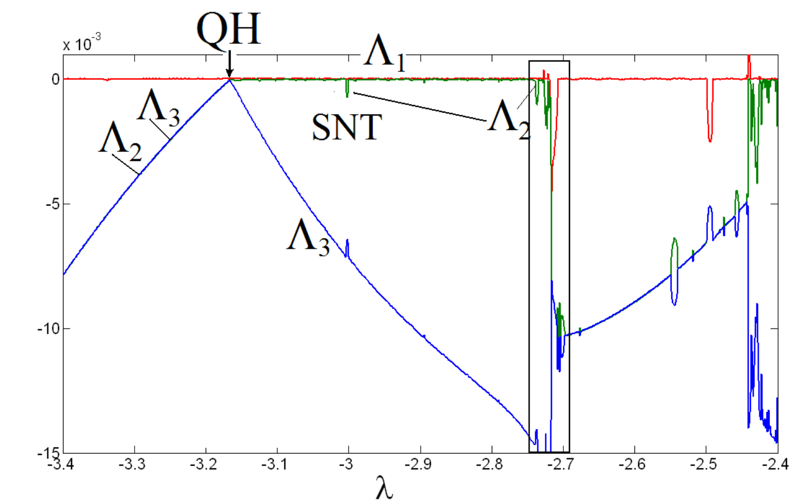
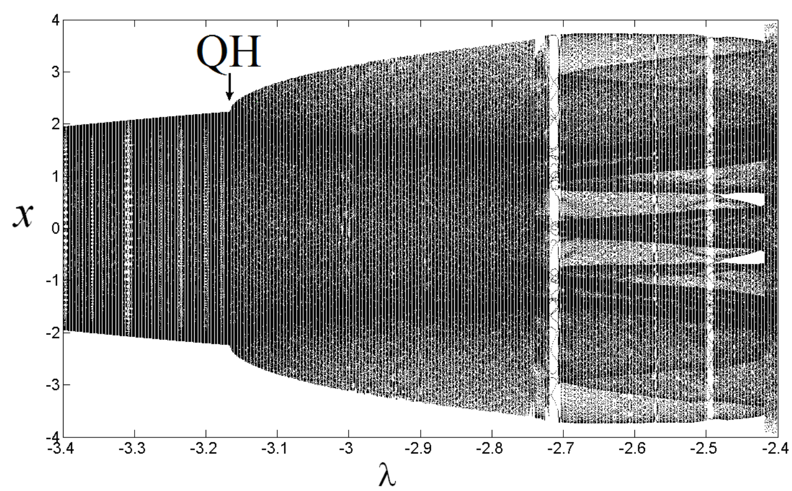


Fig. 5. Attractors of torus map (3). Parameter of discretization $h = 0.1$.



(a)



(b)

Fig. 6. (a) Dependence of three Lyapunov exponents on parameter λ and (b) bifurcation tree, numerically calculated for the model (3). The point of quasi-periodic Hopf bifurcation QH is marked. Discretization step $h = 0.1$, parameter $\beta = 0.081$.

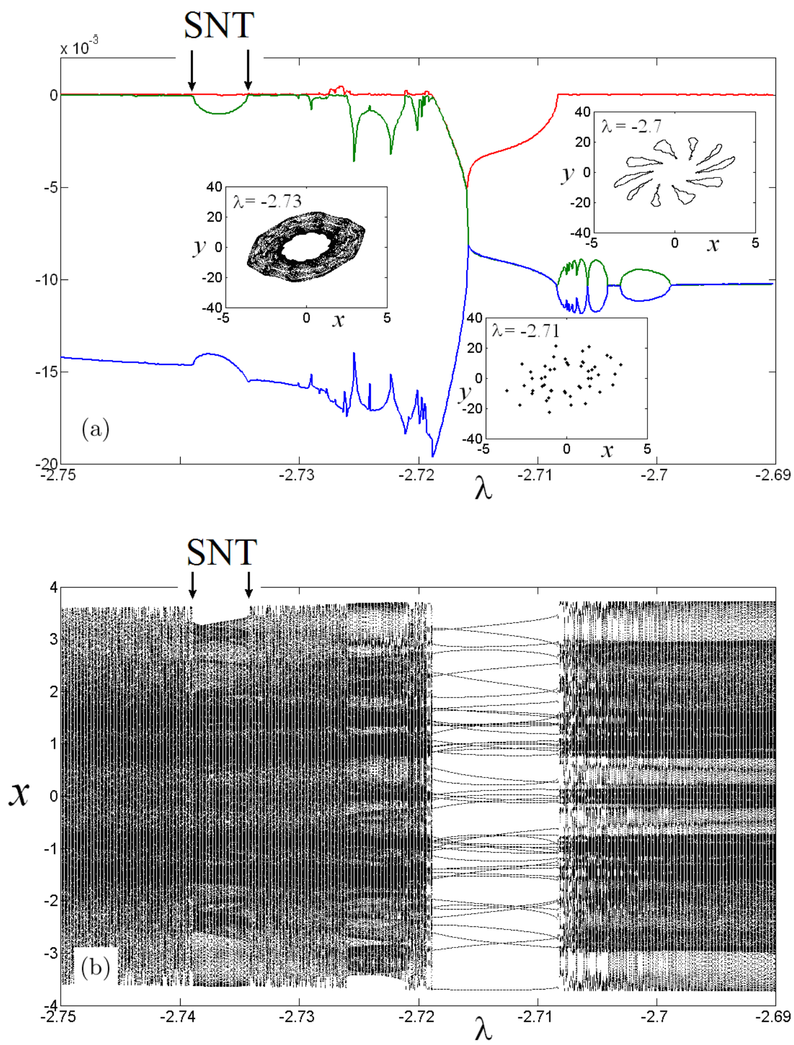


Fig. 7. Enlarged fragment of Fig. 6. Points of saddle-node torus bifurcation SNT and typical phase portraits are indicated.

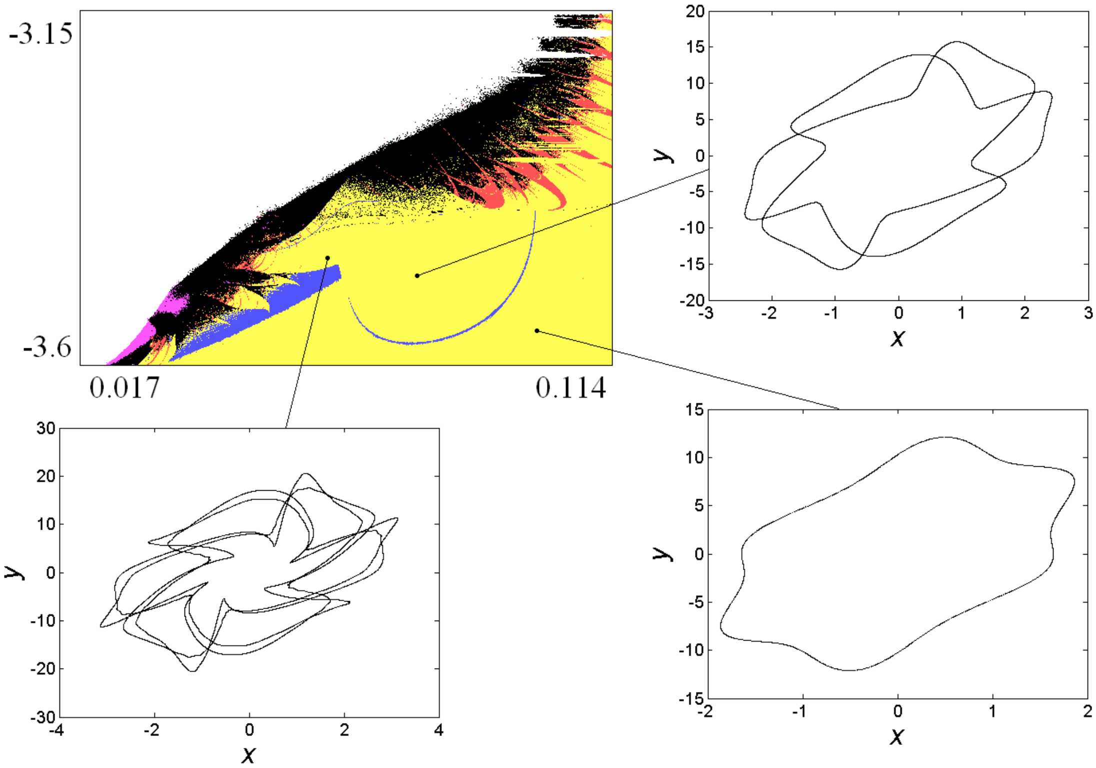


Fig. 8. A chart of Lyapunov exponents and phase portraits of torus map (3) at $h = 0.16$.

is a torus-doubling bifurcation [Anishchenko, 1995; Vitolo *et al.*, 2011]. Indeed, instead of a single oval we have two overlapping ovals. They are obtained by the projection of two isolated closed invariant curves lying in three-dimensional space (x, y, z) on the surface of the torus. The phase portrait in Fig. 8 illustrates another such bifurcation for already doubled torus. Thus, on increasing the discretization parameter, a new transformation occurs in comparison with Fig. 4.

4. Nonautonomous Model with Quasi-Periodic Dynamics

It is interesting to compare the studied dynamics with the case of nonautonomous systems. It is necessary to choose such a system to investigate two-frequency quasi-periodicity and Neimark–Sacker bifurcation. An appropriate example is a universal map [Kuznetsov *et al.*, 2012a] for which there are all main bifurcation scenarios of two-dimensional maps:

$$x_{n+1} = Sx_n - y_n - (x_n^2 + y_n^2),$$
$$y_{n+1} = J + x_n - \frac{1}{5}(x_n^2 + y_n^2). \tag{4}$$

In Fig. 9, we demonstrate a diagram for the universal map [Kuznetsov *et al.*, 2012b], which combines the properties of Lyapunov exponent chart and the chart of periodic regimes. It contains Neimark–Sacker bifurcation line NS, $J = 1$, and a set of Arnold regular tongues.

Let us use the next form of external driving:

$$x_{n+1} = Sx_n - y_n - (x_n^2 + y_n^2),$$
$$y_{n+1} = J + x_n - \frac{1}{5}(x_n^2 + y_n^2) + \varepsilon \cos 2\pi\theta_n, \tag{5}$$
$$\theta_{n+1} = w + \theta_n \quad (\text{mod } 1).$$

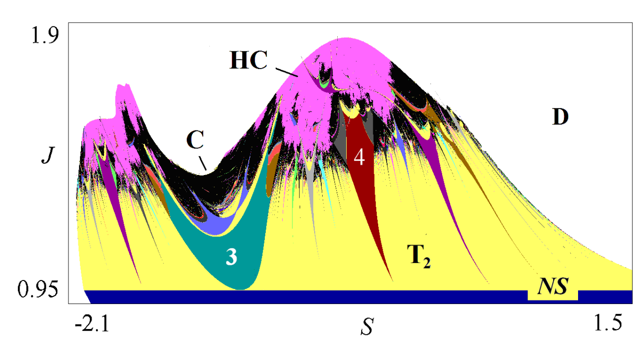


Fig. 9. Combined chart of Lyapunov exponents and chart of periodic regimes for universal two-dimensional map (4), NS — line of Neimark–Sacker bifurcation.

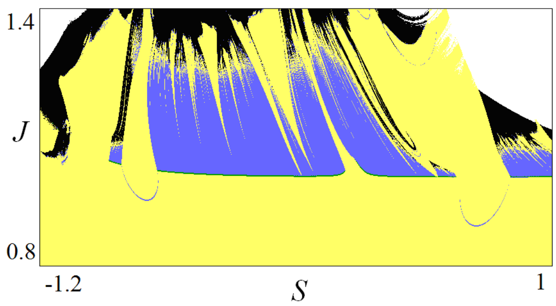


Fig. 10. Lyapunov chart of universal two-dimensional map (5) with quasi-periodic external force. Parameter $\varepsilon = 0.1$.

A parameter determining the frequency of driving force is equal to the golden mean, $w = \frac{\sqrt{5}-1}{2}$, which provides the quasi-periodic type of observed dynamics.

The Lyapunov chart of model (5) is shown in Fig. 10. Compared with Fig. 9 the periodic regimes are replaced by two-frequency regimes, and two-frequency regimes become three-frequency ones in Fig. 10. Also we can see bands of two-frequency modes replacing the tongues as well as typical ovals at the bottom of some of the wide tongues corresponding to torus-doubling bifurcations. However, a significant difference from the dynamics discussed above is the lack of periodic resonances within tongues. Accordingly, there are no secondary two-frequency resonances in their neighborhood.

Thus, the nonautonomous systems exhibit some characteristics that are typical for autonomous models with quasi-periodicity, but fundamental differences are inevitable.

5. Conclusion

Thus, the substitution of time derivatives by finite differences in the equation when generating quasi-periodic oscillations (1) provides a new convenient model in the form of a three-dimensional map. This map demonstrates the regimes of two-frequency and three-frequency quasi-periodicity and all basic quasi-periodic bifurcations. Typical Lyapunov exponent plots and bifurcation trees are presented. Such map allows to study a three-parameter structure of quasi-periodic Hopf bifurcation. With variation of third parameter there are significant changes in the structure of the parameter plane and the form of phase portraits. The fundamental difference from nonautonomous systems with quasi-periodic forcing is the possibility of various periodic resonances and the secondary tongues of two-frequency regimes associated with them.

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