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ELECTRODYNAMICS AND WAVE PROPAGATION

High-Q Resonanaces of Surface Waves in Thin Metamaterial Cylinders

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Abstract—The 2D problem of excitation of a circular metamaterial cylinder by a filament source is numerically investigated. It is found that, when the relative permittivity and permeability are close to minus unity, high-Q resonances occur in cylinders of an electrically small diameter. Near- and far-field patterns are calculated. It is discovered that, under resonance conditions, a multilobe scattering pattern typical of superdirective antennas is formed. The influence of loss on the resonance characteristics is investigated.

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INTRODUCTION

Recently, the electromagnetics of artificial media characterized by negative relative permittivity and permeability ε and μ has intensely developed. Veselago is the first to theoretically investigate the radiophysical properties of such media [1], which are often called Veselago media. Today, these artificial media are conventionally called metamaterials [2].

The interaction between the electromagnetic field and metamaterial objects is accompanied by a number of extraordinary effects: negative refraction, subwavelength resolution, and the inversion of the Doppler and Vavilov–Cerenkov effects. These effects are investigated in a great number of studies. Note reviews [3–6].

The modified discrete source method [7, 8] makes it possible to effectively calculate wave fields in 2D problems of diffraction by magnetodielectric bodies of complex shapes. In most studies, objects of dimensions that are large as compared to the wavelength are investigated. The electrodynamic properties of small 2D objects were analyzed for the first time in [9, 10].

The purpose of this study is to investigate the resonance properties of an electrically small cylinder made from a metamaterial with permittivity ε and permeability μ that are close to minus unity.

1. FORMULATION OF THE PROBLEM

The problem of excitation of a metamaterial cylinder with parameters $\varepsilon < 0$ and $\mu < 0$ by a filament source is considered. The case of the TM polarization is studied. It is assumed that the source is situated beyond the cylinder at the point $r_0 > a$, $\varphi_0 = 0$ (Fig. 1). Cylindrical coordinates (r, φ, z) are used.

The problem formulated above is reduced to determination of scalar function $U(r,\varphi) = H_z(r,\varphi)$ satisfying the inhomogeneous Helmholtz equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + k^2\varepsilon(r)\mu(r)\right]U(r,\varphi) = -\frac{4i}{r}\delta(r - r_0)\delta(\varphi), \tag{1}$$

where k is the wave number in free space, functions $\varepsilon(r)$ and $\mu(r)$ are specified as

$$\varepsilon(r) = \begin{cases}\varepsilon, & 0 < r < a,\\ 1, & r > a,\end{cases} \qquad \mu(r) = \begin{cases}\mu, & 0 < r < a,\\ 1, & r > a,\end{cases} \tag{2}$$

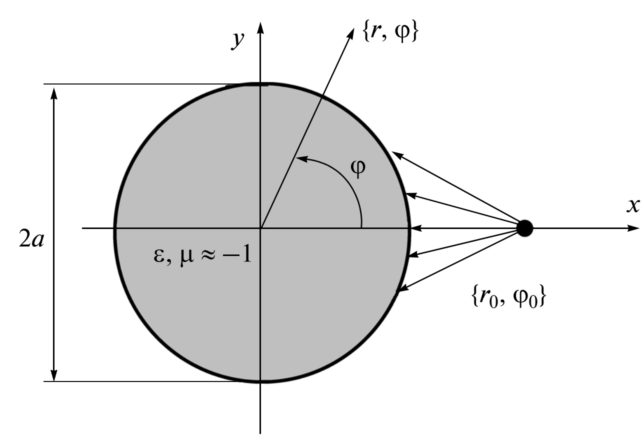

Fig. 1. Geometry of the problem.

and δ(...) is the Dirac delta function.

The conditions

$$U(a-0,\varphi)=U(a+0,\varphi),\quad \frac{1}{\varepsilon}\frac{\partial U}{\partial r}(a-0,\varphi)=\frac{\partial U}{\partial r}(a+0,\varphi) \tag{3}$$

are fulfilled on the boundary $r=a$. Field $U(r,\varphi)$ satisfies the radiation conditions, i.e., has the form

$$U(r,\varphi)=\Phi(\varphi)(2/\pi kr)^{1/2}\exp\left(-ikr+i\frac{\pi}{4}\right) \tag{4}$$

as $kr\to\infty$, where $\Phi(\varphi)$ is the pattern.

The field of the incident cylindrical wave is a solution to Eq. (1) at $\varepsilon=1$ and $\mu=1$. This field is determined from the formula

$$U^0(r,\varphi)=H_0^{(2)}\left(k\sqrt{r^2+r_0^2-2rr_0\cos\varphi}\right), \tag{5}$$

where $H_0^{(2)}$ is the Hankel function. Field pattern $U^0(r,\varphi)$ has the form

$$\Phi^0(\varphi)=\exp(ikr_0\cos\varphi). \tag{6}$$

Thus, with the chosen normalization of the right-hand side of Eq. (1), wave field $U(r,\varphi)$ is a dimensionmless quantity.

Consider diffraction phenomena in the case when the dimensions of the cylinder are substantially smaller than the wavelength $\lambda=2\pi/k$. For cylinders of large electric dimensions ($ka\gg1$), this problem is numerically investigated in [11] with the help of the modified discrete source method.

2. THE METHOD OF SOLUTION

The diffraction problem formulated above, can be analytically solved by means of the method of separation of variables (the Rayleigh series [12]). For cylinders of small electric dimensions, this series can effectively be applied in the numerical calculation. Let us present the basic formulas of the Rayleigh method.

The field in the interior of the cylinder ($r\le a$) can be decomposed as follows:

$$U(r,\varphi)=-\frac{2i}{\pi ka}\times\sum_{m=0}^{\infty}\frac{\delta_m H_m^{(2)}(kr_0)J_m(knr)\cos(m\varphi)}{H_m^{(2)\prime}(ka)J_m(kna)-\frac{n}{\varepsilon}H_m^{(2)}(ka)J'_m(kna)}, \tag{7}$$

where

$$\delta_m=\begin{cases}1, & m=0,\\ 2, & m\ge1,\end{cases} \tag{8}$$

$$n=\sqrt{\varepsilon\mu}, \tag{9}$$

J_m are Bessel functions, and the prime denotes differentiation with respect to an argument.

The field in the exterior of the cylinder ($r\ge a$) consists of two terms (the incident and scattered fields):

$$U(r,\varphi)=U^0(r,\varphi)+U^s(r,\varphi), \tag{10}$$

where the scattered field is determined from the formula

$$U^s(r,\varphi)=-\sum_{m=0}^{\infty}\frac{\delta_m H_m^{(2)}(kr_0)\left[J'_m(ka)J_m(kna)-\frac{n}{\varepsilon}J_m(ka)J'_m(kna)\right]H_m^{(2)}(kr)\cos(m\varphi)}{H_m^{(2)\prime}(ka)J_m(kna)-\frac{n}{\varepsilon}H_m^{(2)}(ka)J'_m(kna)}. \tag{11}$$

Far field U^s ($kr\to\infty$) has the form

$$U^s(r,\varphi)\sim\Phi^s(\varphi)(2/\pi kr)^{1/2}\exp(-ikr+i\pi/4), \tag{12}$$

where scattering pattern $\Phi^s(\varphi)$ can be represented in the form of the series

$$\Phi^s(\varphi)=-\sum_{m=0}^{\infty}\frac{\delta_m(i)^m H_m^{(2)}(kr_0)\left[J'_m(ka)J_m(kna)-\frac{n}{\varepsilon}J_m(ka)J'_m(kna)\right]\cos(m\varphi)}{H_m^{(2)\prime}(ka)J_m(kna)-\frac{n}{\varepsilon}H_m^{(2)}(ka)J'_m(kna)}. \tag{13}$$

Formulas (7), (11), and (13) contain quantity $n=\sqrt{\varepsilon\mu}$, which formally coincides with the refractive index of a plane wave propagating in a homogeneous medium with constitutive parameters ε and μ. As is known, the refractive index of plane waves propagating in a homogeneous medium with tensor permittivity $\hat{\varepsilon}$ and permeability $\hat{\mu}$ can be found from the Fresnel biquadratic equation [13]. In the case of an isotropic medium, the Fresnel equation is reduced to the quadratic equation $n^2=\varepsilon\mu$, which has two solutions $n=\pm\sqrt{\varepsilon\mu}$. In the electromagnetics of ordinary media ($\varepsilon>0$ and $\mu>0$), the plus sign is conventionally chosen. Some authors recommend choosing the minus sign for describing Veselago media ($\varepsilon<0$ and $\mu<0$). This choice of the sign is suitable, for example, when Snell's law is used. However, it should be kept in mind that, strictly speaking, the term *refractive index* characterizes the properties of plane waves propagating in a medium rather than the properties of this medium. The use of the terminology involving refractive indices is not always effective and can yield incorrect results in certain cases. It is more suitable to use the notions of permittivity and permeability, which are the only constitutive parameters entering Maxwell's equations [5].

As should have been expected, formulas (7), (11), and (13) are invariant to the sign of n. Helmholtz equation (1) contains the quantity $n^2 = \varepsilon\mu$. Actually, the numerators and denominators of individual terms from decompositions (7), (11), and (13) are simultaneously either even or odd functions of parameter n. This fact follows from the formulas

$$J_m(-x) = (-1)^m J_m(x), \quad J'_m(-x) = -(-1)^m J'_m(x). \quad (14)$$

3. NUMERICAL RESULTS

The numerical results presented below are obtained both with the help of the modified discrete source method [11] and from formulas (7), (11), and (13). The results of the corresponding computations are in good agreement. In all of the illustrations, we describe the spatial structure of the field on the linear scale and the frequency characteristics of the field on the logarithmic scale.

First, we investigate the dependence of the absolute value of the total field at the point $r = a$, $\varphi = \pi$ located on the shadow side of the cylinder on parameter ka, which is proportional to frequency; i.e., we investigate the amplitude–frequency characteristic (AFC) of the total field. We disregard the frequency dispersion of the metamaterial. In all of the computation, the spatial coordinate of the cylindrical wave source is set to be $r_0 = 1.2a$. We have found out that, in the case

$$\varepsilon = -1.01, \quad \mu = -0.91 \quad (15)$$

the frequency characteristic is a sequence of resonance peaks. The resonance frequencies of the first eight modes are summarized in the ascending order in the table. We restrict the analysis to lower order resonances that are low-frequency ones. It is shown below that the resonance frequencies are enumerated so that, at resonance frequency ka_m, scattering pattern $\Phi^s(\varphi)$ and field $U(a,\varphi)$ on the cylinder's surface are described by one azimuthal harmonic $\cos(m\varphi)$ with a high accuracy.

Figure 2 shows the AFC with two resonance peaks at frequencies ka_3 and ka_4 calculated from formula (7). The Q factors of these resonances are evaluated by the respective values $Q_3 = 150$ and $Q_4 = 1050$. It is seen that the Q factor of the resonances grows with frequency.

The scattering patterns calculated at frequencies ka_3 and ka_5 are displayed in Figs. 3 and 4. The patterns from Figs. 3 and 4 contain six and ten identical lobes, respectively. Note that, for the scatterer's dimension $2a \ll \lambda$, these lobes are characterized by rather small angular dimensions. This means that the superdirectivity effect occurs under resonance conditions. In addition, the resonance amplitude of the scattering pattern substantially exceeds the amplitude of the incident field pattern: $|\Phi^s(\varphi)| \gg 1$.

Resonances of surface waves in a metamaterial cylinder with the parameters $\varepsilon = -1.01$, and $\mu = -0.91$

Resonance number m	Frequency ka_m	Frequency interval $(ka_{m+1} - ka_m)$
3	0.455450	0.22429
4	0.707479	0.25203
5	0.980963	0.27348
6	1.272137	0.29117
7	1.578155	0.30602
8	1.896774	0.31862
9	2.226206	0.32943
10	2.565006	0.33880

The scattering pattern calculated at the nonresonance frequency $ka = 0.844$, i.e., at a frequency lying between resonance frequencies ka_4 and ka_5, is displayed in Fig. 5. It is seen that, in this case, the scattered field is practically omnidirectional, and scattering pattern Φ^s is commensurable with incident field pattern Φ^0.

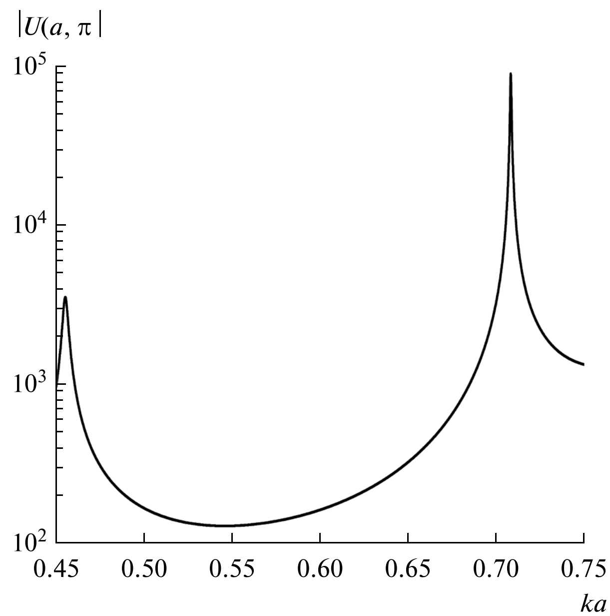

Fig. 2. Absolute value of field $|U(a,\pi)|$ in the neighborhood of frequencies ka_3 and ka_4.

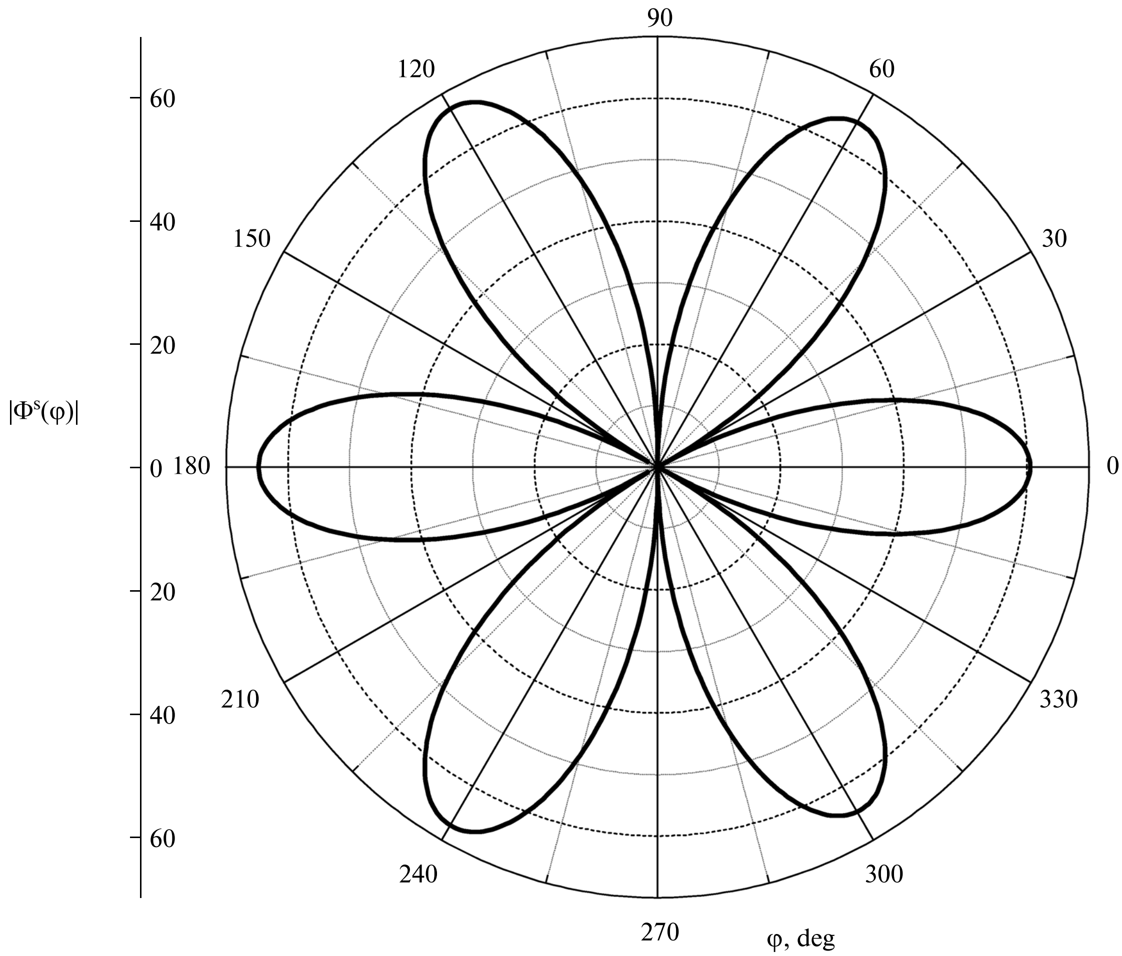

Fig. 3. Absolute value of scattering pattern $\left|\Phi^s(\varphi)\right|$ at the resonance frequency $ka_3 = 0.455$.

Figure 6 shows the field on the cylinder's surface at resonance frequency ka_5. This field is described by the function $\cos(5\varphi)$ with the graphical accuracy. The field at resonance frequency ka_3 is depicted in Fig. 7. This field is described by the function $\cos(3\varphi)$. Thus, from Figs. 3, 4, 6, and 7, we can draw the conclusion that, at resonance frequencies ka_3 and ka_5, one term of the series with $m = 3$ or $m = 5$ dominates in decompositions (7), (11), and (13). We should emphasize that the amplitudes of resonance harmonics $\cos(m\varphi)$ are very large in the near field (on the order 10^4–10^3) and remain rather large in the far field ($\Phi^s(\varphi)$ being on the order 10^2, see Figs. 3 and 4). Note that the absolute value of the incident field pattern $\Phi^0(\varphi)$ is unity.

The near-field pattern at the nonresonance frequency $ka = 0.844$ has a more intricate form (Fig. 8). The amplitude of field oscillations on the illuminated side of the cylinder is smaller than that on the shadow side. The field decreases by three orders of magnitude as compared to the resonance case ka_5.

The dependence of the field of resonance oscillation ka_5 on radial coordinate r is depicted in Fig. 9. It is seen that function $U(r, \pi)$ is concentrated within a narrow interval near the surface $r = a$. This localization of the field is typical of surface waves propagating over a plane boundary of a metamaterial [14]. The abrupt increase of the field at certain frequencies is related with the resonances of a surface wave propagating over the interface $r = a$. Resonances occur when the phase of the surface wave covering the total length of the cylinder's circle increases by the value $2\pi m$, a situation that is possible for small cylinders in the case of an extremely slow wave.

Figure 10 illustrates the influence of loss on the APC in the neighborhood of the resonance frequency ka_3. The absorbing properties of a medium with complex refractive index n are characterized by the quantity $\nu = -\mathrm{Im}(n)$. It is seen that, as the loss grows, the

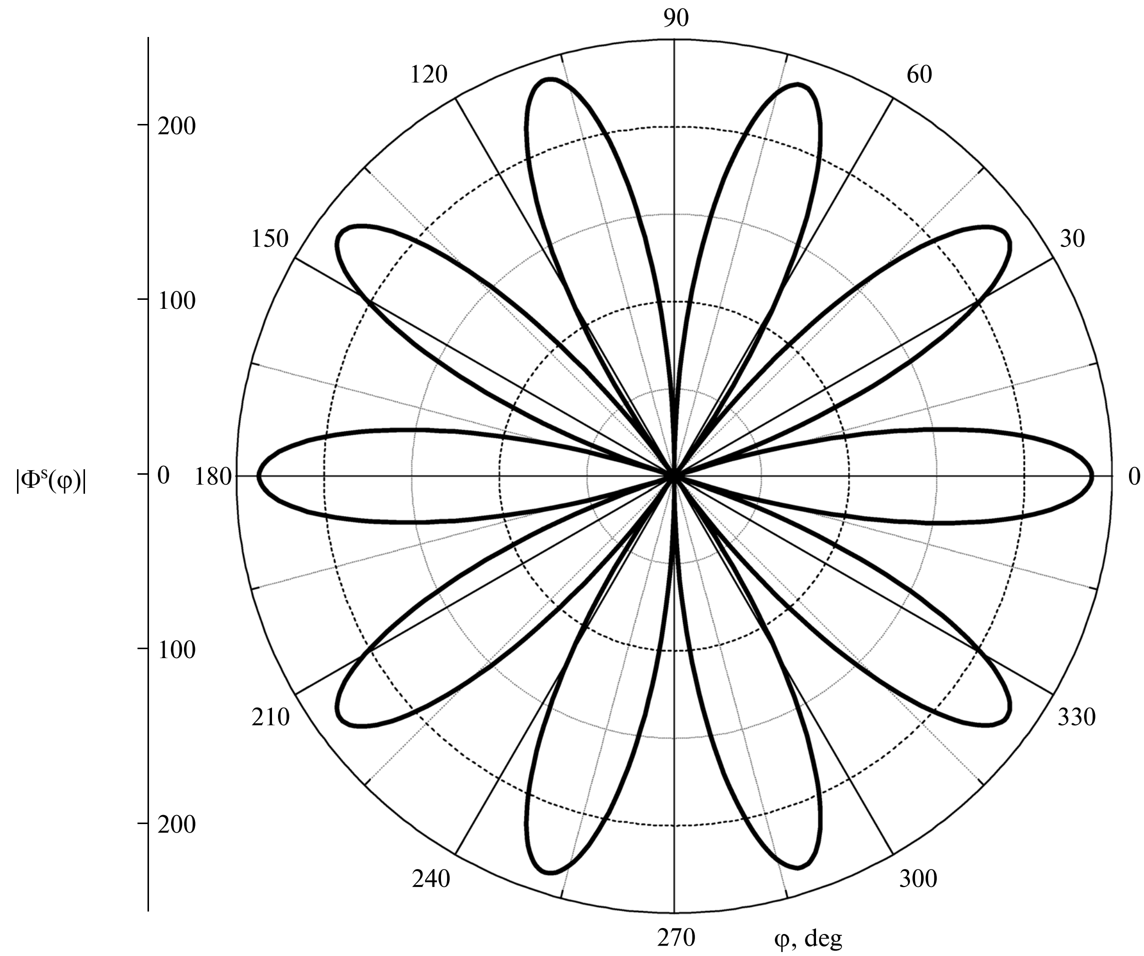

Fig. 4. Absolute value of scattering pattern $\left|\Phi^s(\varphi)\right|$ at the resonance frequency $ka_5 = 0.981$.

resonance Q factor decreases and the resonance frequency shifts toward low frequencies.

For the resonance frequencies satisfying the condition

$$ka_m \ll 1, \quad (16)$$

simple approximate analytic expressions can be obtained. Formulas (7), (11), and (13) contain the resonance denominator

$$H_m^{(2)\prime}(ka)J_m(kna) - \frac{n}{\varepsilon}H_m^{(2)}(ka)J_m'(kna). \quad (17)$$

Let us investigate the frequency dependence of this denominator assuming that

$$ka \ll 1, \quad kna \ll 1; \quad (18)$$

$$\varepsilon \to -1. \quad (19)$$

Expression (17) is a complex function of parameter ka and does not vanish at real values of ka. Note that, under conditions (18), the imaginary part of expression (17) substantially exceeds its real part. The imaginary part of denominator from (17) vanishes at the points ka_m that are the sought resonance frequencies. To determine these frequencies, we apply the known asymptotic decompositions of cylindrical functions for small values of the argument. We use two terms of the decomposition in the positive powers of the argument for Bessel functions and two terms of the decomposition in the negative powers of the argument for Hankel functions. Taking into account condition (19), we obtain the sought expression for resonance frequencies

$$(ka_m)^2 = \frac{2m(m^2-1)(\varepsilon+1)}{\mu - 1 - m(\mu+1)}, \quad m \geq 2. \quad (20)$$

As it follows from formula (20), the main condition for the existence of low-frequency resonances is the closeness of the permittivity to minus unity (condition (19)). This condition ensures the fulfillment of inequality (16) (at least, for the values of index m that are not very large).

For expression (20) to be positive, it is necessary to impose a constraint on the range of ε and μ. To this

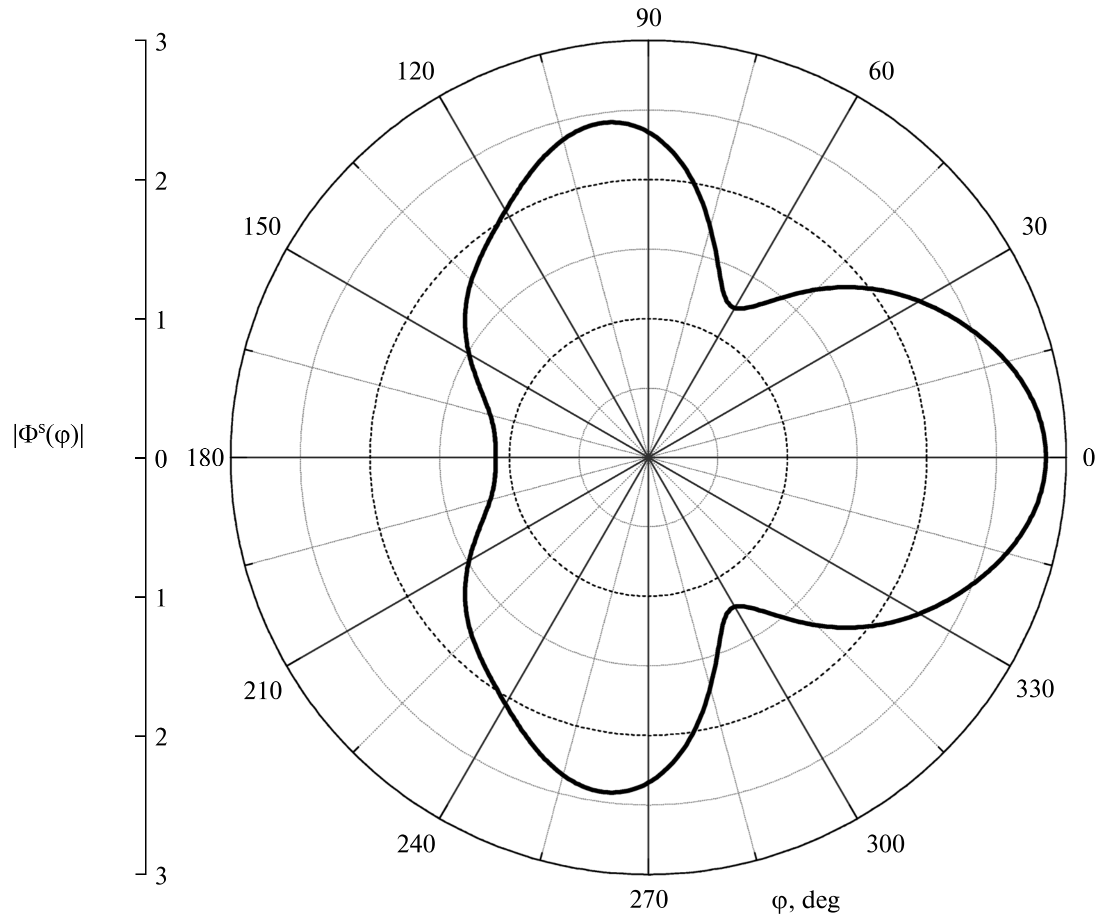

Fig. 5. Absolute value of scattering pattern $|\Phi^s(\varphi)|$ at a frequency lying between the fourth and fifth resonances ($ka = 0.844$).

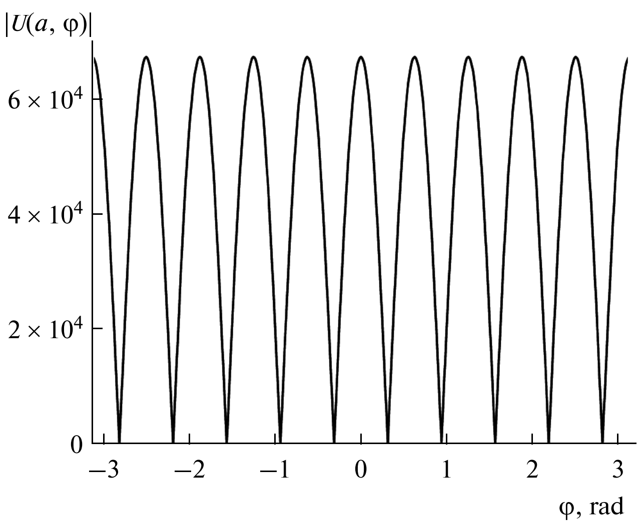

Fig. 6. Absolute value of field $|U(a, \varphi)|$ on the cylinder's surface at resonance frequency ka_5.

end, any pair of the inequalities presented below should be fulfilled:

$$\varepsilon + 1 < 0, \quad \mu - 1 - m(\mu + 1) < 0; \tag{21}$$

$$\varepsilon + 1 > 0, \quad \mu - 1 - m(\mu + 1) > 0. \tag{22}$$

In contrast to the permittivity, which should be close to minus unity, the permeability can change within a wide range. For example, the parameters $\mu = -1$ and $\varepsilon < -1$ satisfy condition (21). Then,

$$(ka_m)^2 = -m(m^2 - 1)(\varepsilon + 1). \tag{23}$$

Plasma with the parameters $\mu = 1$ and $\varepsilon < -1$ satisfies condition (21). The resonance frequency is determined from the formula

$$(ka_m)^2 = -(m^2 - 1)(\varepsilon + 1). \tag{24}$$

Parameters (15), which are used in the numerical calculation, also satisfy inequalities (21). Inequalities (22) are fulfilled, for example, for the parameters $\varepsilon = -0.99$ and $\mu = -3$.

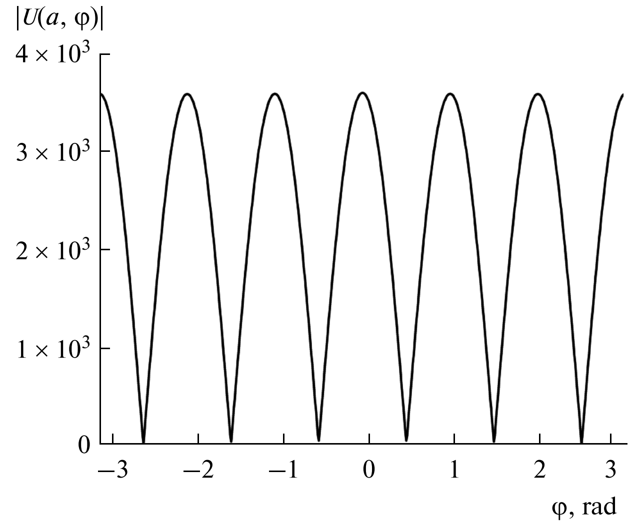

Fig. 7. Absolute value of field $|U(a, \varphi)|$ on the cylinder's surface at resonance frequency ka_3.

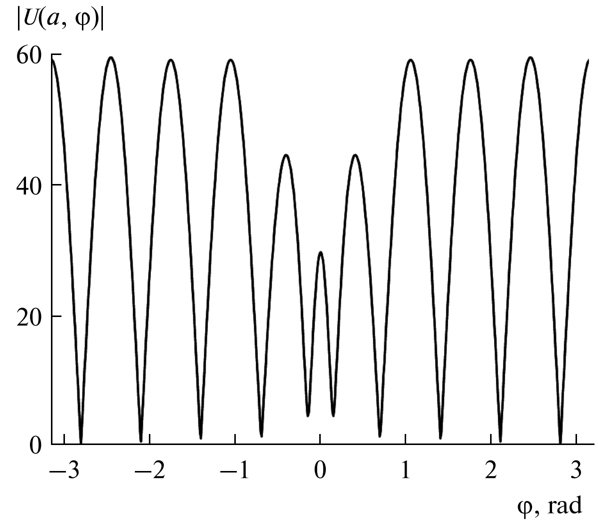

Fig. 8. Absolute value of field $|U(a, \varphi)|$ on the cylinder's surface at a frequency lying between the fourth and fifth resonances ($ka = 0.844$).

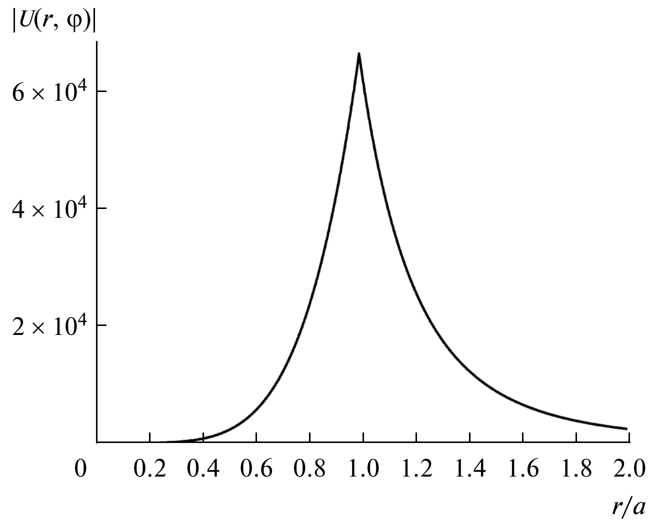

Fig. 9. Radial distribution of field $|U(r, \pi)|$ at resonance frequency ka_5.

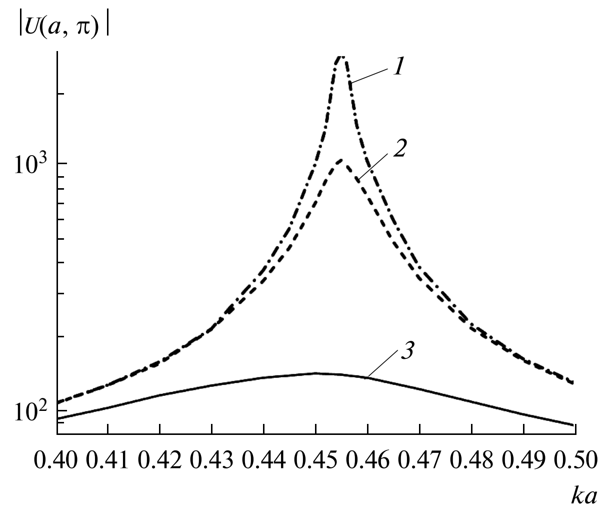

Fig. 10. Absolute value of field $|U(a, \pi)|$ in the neighborhood of resonance frequency ka_3 for various values of the loss factor. Curves *1*, *2*, and *3* correspond to $\nu = 0.001$, 0.01, and 0.1, respectively.

For constitutive parameters (15) and $m = 3$, Eq. (20) yields $ka_3 = 0.469\ldots$ This quantity is in good agreement with the rigorous value $ka_3 = 0.455\ldots$

A solution to the problem of excitation of a metamaterial cylinder by an electric-current filament can be obtained from the above formulas with the help of the following change of notation: $U(r,\varphi) \to E_z(r,\varphi)$, $\varepsilon \to \mu$, and $\mu \to \varepsilon$. Therefore, in the case of the TE polarization, low-frequency resonances occur when the permeability approaches minus unity.

CONCLUSIONS

Resonances can also exist in thin cylinders ($ka \ll 1$) made from a standard magnetodielectric with a large refractive index. In 2D electromagnetics, other objects exhibiting resonance properties in the low-frequency region are known: a hollow metal cylinder with a narrow longitudinal slot and a multiturn wire helix with a large helix angle. Note that these structures contain metal, i.e., a material with the large absolute value of a complex permittivity. In the aforementioned examples, volume resonance oscillations develop, and the field occupies the entire interior cavity of a cylinder. The case considered in the study differs from these analogs by the fact that the refractive index of the metamaterial forming the cylinder is not large (being close to unity) and the resonance field has a fundamentally different spatial structure: it is concentrated within a narrow strip near the cylinder's boundary. The investigated structure can be regarded as a high-Q ring resonator operating with the use of an extremely slow surface wave.

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