Characteristics of the modes of planar W-lightguides with arbitrary contrast of the refractive-index profile

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(Submitted May 27, 2013)
Opticheskiĭ Zhurnal **81**, 59–65 (January 2014)

The mode characteristics of a planar five-layer W-lightguide with various refractive indices of the intermediate layer are investigated, based on a numerical solution of the dispersion equation. It is shown that it is easy to ensure the single-mode regime in a broad range of wavelengths, the necessary field concentration in a large-diameter lightguide core, and efficient filtering of the evanescent modes in W-lightguides by appropriately choosing the geometrical and optical parameters. *© 2014 Optical Society of America.*

OCIS codes: (060.2400) Fiber properties; (130.2790) Guided waves; (230.7390) Waveguides, planar.
http://dx.doi.org/10.1364/JOT.81.000044

INTRODUCTION

There currently is interest in optical waveguides with a W profile[1–4] because there is a prospect of widely using them in various devices where there is a requirement of a single-mode regime in a specified wavelength range, a large cross section of the fundamental mode (and, as a result, a lower power density of the radiation in the core), and also low optical losses of radiation at a bend. The last condition can also be satisfied with appropriate falloff (loss) of radiation density in the intermediate layer of the W-lightguide (between the core and the outer cladding).

The above can be achieved by correctly choosing the design of the W-lightguide, which depends on at least five parameters: the three values of refractive index n_1, n_2, and n_3 ($n_1 > n_3 > n_2$) and the a and b values (Fig. 1). A W-lightguide with the required parameters can be created by the widely used MCVD technology, which under modern conditions makes it possible to create a lightguide with a specified structure and refractive-index contrast. For example, a W-lightguide can be fabricated with increased contrast, i.e., a relatively large $(n_1 - n_2)$ difference and at the same time a small $(n_1 - n_3)$ refractive-index difference of the layers, controlled to the necessary extent (at a level of about 10^{-3}).[1]

As follows from the calculations, the W configuration makes it easier to meet the contrast requirements indicated above.

The photonic-crystal waveguides that are widely used at present (see, for example, Skibina *et al.*'s review[5]), which include microstructured lightguides and lightguides with a Bragg structure, possess a rich spectrum of properties and make it possible, among other things, to obtain large-diameter guided modes (with a core size of about 40λ or larger[6]). However, they have still higher sensitivity to bending and a fairly complex fabrication technology. In turn, W-lightguides possess more limited possibilities than photonic-crystal waveguides but are simpler and consequently more easily fabricated. At the same time, in a number of cases in which nonlinear processes are undesirable or, on the other hand, when it is necessary to increase their efficiency, W-lightguides can be much more efficient for certain applications[3] than (at least) standard lightguides with a stepped refractive index, since they promise to be less sensitive to bending.

This paper is devoted to an investigation of the main characteristics of the modes in a planar five-layer W-lightguide, as a function of the contrast of the refractive- index profile. The results obtained by the authors in Ref. 2 will be used, particularly for the numerical calculations and the construction of graphs.

THEORY

It is assumed (see Fig. 1) that the values of n_1, n_2, and n_3 are given, as well as those of a and b (in micrometers), and that the following condition is satisfied:

$$n_1 > n_3 > n_2. \quad (1)$$

The only component $(\partial/\partial y = 0)$ of the electric field of the TE mode has the form

$$E_y = E(x)\exp(-i\beta z) \quad (2)$$

and satisfies the wave equation

$$\frac{d^2E(x)}{dx^2} + (k^2n^2(x) - \beta^2)E(x) = 0, \quad (3)$$

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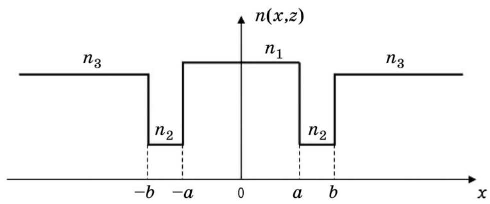

FIG. 1. Schematic illustration of the transverse refractive-index distribution in a planar five-layer light-conducting structure.

where $\beta = \beta' - i\beta''$ is the propagation constant of the mode, and $k = \omega/c$.

Using the continuity conditions of the functions $E_y \sim E(x)$ and $H_z \sim dE(x)/dx$ at points $x = a$ and $x = b$, the solution of Eq. (3) and the dispersion equation for finding the wave numbers of the modes can be represented, respectively, in the form of Eqs. (4) and (6):

$$E(x) = \begin{cases} A\left[1 + \frac{w-v}{w+v}\exp[-2w(b-a)]\right] f(x) & 0 \le x \le a, \\ A\left[1 + \frac{w-v}{w+v}\exp[-2w(b-x)]\right]\exp[-w(x-a)] & a \le x \le b, \\ A\left[1 + \frac{w-v}{w+v}\right]\exp[-w(b-a)]\exp[-v(x-b)] & x \ge b, \end{cases} \tag{4}$$

where $f(x) = \cos(ux)/\cos(ua)$ for even modes and $f(x) = \sin(ux)/\sin(ua)$ for odd modes; u, w, and v are the transverse wave numbers

$$\begin{aligned} u^2 &= k^2 n_1^2 - \beta^2, \\ w^2 &= \beta^2 - k^2 n_2^2, \\ v^2 &= \beta^2 - k^2 n_3^2, \end{aligned} \tag{5}$$

$$F(v) = wC\cos(ua) - uD\sin(ua) = 0 \quad \text{for even modes,} \tag{6a}$$

$$F(v) = wC\sin(ua) + uD\cos(ua) = 0 \quad \text{for odd modes.} \tag{6b}$$

Here

$$\begin{aligned} C &= v\cosh[w(b-a)] + w\sinh[w(b-a)], \\ D &= w\cosh[w(b-a)] + v\sinh[w(b-a)]. \end{aligned} \tag{7}$$

The internal transverse wave numbers u and w that appear in Eqs. (4), (6), and (7) should be regarded as functions of variable v (the external transverse wave number v), which are determined from

$$\begin{aligned} u^2 &= -v^2 + k^2(n_1^2 - n_3^2), \\ w^2 &= v^2 + k^2(n_3^2 - n_2^2), \end{aligned} \tag{8}$$

which follow from Eq. (5).

The roots of Eq. (6) can be computed by Newton's method:[7]

$$v_{m+1} = v_m - \frac{F(v_m)}{F'(v_m)}. \tag{9}$$

It follows from Eq. (7) that

$$\frac{dw}{dv} = \frac{v}{w}, \qquad \frac{du}{dv} = -\frac{v}{u}. \tag{10}$$

Using Eq. (10), we get from Eqs. (6) and (7) an explicit expression for the derivative of the function $F(v)$:

$$\begin{aligned} F'(v) &= C\left\{\frac{v}{w}\cos(ua) + \left[-\frac{u}{w} + \frac{v(w^2+u^2)}{uw}a - \frac{uv}{w}b\right]\sin(ua)\right\} \\ &\quad + D\left\{(1+vb)\cos(ua) + \frac{u}{v}\sin(ua)\right\} \quad \text{for even modes,} \end{aligned} \tag{11a}$$

$$\begin{aligned} F'(v) &= C\left\{\frac{v}{w}\sin(ua) - \left[-\frac{u}{w} + \frac{v(w^2+u^2)}{uw}a - \frac{uv}{w}b\right]\cos(ua)\right\} \\ &\quad + D\left\{(1+vb)\sin(ua) - \frac{u}{v}\cos(ua)\right\} \quad \text{for odd modes.} \end{aligned} \tag{11b}$$

RESULTS OF THE CALCULATIONS

We recall that undamped directed waves correspond to the actual positive values of u, w, and v that are found by solving dispersion Eq. (6), using Eq. (8). The complex roots $v = v' + iv''$ of the same Eq. (6) with a negative real part $v' < 0$ correspond to evanescent waves whose losses are calculated from

$$\alpha = 20\log(e) \times 10^6 \beta'' \ \text{dB/m}, \tag{12}$$

where β'' is substituted numerically in μm^{-1}. Finally, when $v = 0$, dispersion Eq. (6) transforms into an equation for determining the critical frequencies of the W-lightguide[2]

$$\frac{\sqrt{n_3^2 - n_2^2}}{\sqrt{n_1^2 - n_3^2}}\tanh\left[k\sqrt{n_3^2 - n_2^2}(b-a)\right] = \tan\left[ka\sqrt{n_1^2 - n_3^2}\right] \quad \text{for even modes,} \tag{13a}$$

$$\frac{\sqrt{n_3^2 - n_2^2}}{\sqrt{n_1^2 - n_3^2}}\tanh\left[k\sqrt{n_3^2 - n_2^2}(b-a)\right] = \cot\left[ka\sqrt{n_1^2 - n_3^2}\right] \quad \text{for odd modes.} \tag{13b}$$

It is obvious that the parameters that determine the characteristic properties of a W-lightguide are the refractive index n_2 of the intermediate layer and its width $(b-a)$.

Using Eqs. (13a) and (13b), we find the limits of the single-mode and two-mode regimes of certain configurations of W-lightguides. Figure 2(a) shows the calculated dependence of the cutoff frequency of the fundamental mode (the lower limit of the shaded zone), the dependences of the cutoff frequency of the first, odd mode (the upper limit of the shaded zone), and the cutoff frequency of the second, even mode (upper curve) on the refractive index n_2 of the intermediate layer ($1 \leq n_2 \leq n_3$) of the W-lightguide with fixed values of the parameters ($n_1 = 1.456$, $n_3 = 1.453$, $a = 9.5$ μm, and $b = 10$ μm). The points that lie in the darkened zone correspond to the single-mode regime, which has zero cutoff of the fundamental mode in the interval $1.395 < n_2 < n_3$, and the indicated regime is implemented in the maximum frequency range at the limit of the interval (in this case, when $n_2 = 1.395$). The straight line parallel to the horizontal axis at the level $\omega/c \approx 4.054\ \mu\text{m}^{-1}$ corresponds to radiation with wavelength $\lambda = 1.55$ μm. As can be seen from the figure, the given W-lightguide has two modes in a wide range of variation of the refractive index ($1 < n_2 < 1.445$) at that wavelength, whereas it changes to three-mode when $1.445 < n_2 < n_3$.

Let us vary the width of the intermediate layer of the lightguide under consideration while satisfying the condition that the cutoff frequency of the fundamental mode remains zero in the entire interval of variation of n_2 ($1 \leq n_2 \leq n_3$). This requires the use of the inequality that follows from the requirement that the right-hand side of Eq. (13a) is greater than or equal to the left-hand side no matter how small the value of k is,

$$a \geq (b - a)\frac{n_3^2 - n_2^2}{n_1^2 - n_3^2}. \tag{14}$$

It is obvious that, if this inequality is satisfied for some value $\bar{n}_2$, it is even more strongly satisfied for larger values: $\bar{n}_2 \leq n_2 \leq n_3$. We rewrite inequality (14) in the form

$$a \geq b\frac{n_3^2 - n_2^2}{n_1^2 - n_2^2}. \tag{15}$$

It follows from this inequality that, when $n_1 = 1.456$, $n_3 = 1.453$, $b = 10$ μm, and $n_2 = 1$, parameter a (the half-width of the lightguide core) must be about $9.922\ \mu\text{m} \leq a \leq b$. Let $a = 9.93$ μm, and then the limits of the single-mode regime of a W-lightguide take the form shown in Fig. 2(b). As expected, a lightguide was obtained with zero cutoff of the fundamental mode on the entire interval $1 \leq n_2 \leq n_3$. At a wavelength of 1.55 μm, the given W-lightguide has two modes in the interval $1 < n_2 < 1.369$, whereas it acquires three modes when $1.369 < n_2 < n_3$. At a wavelength of

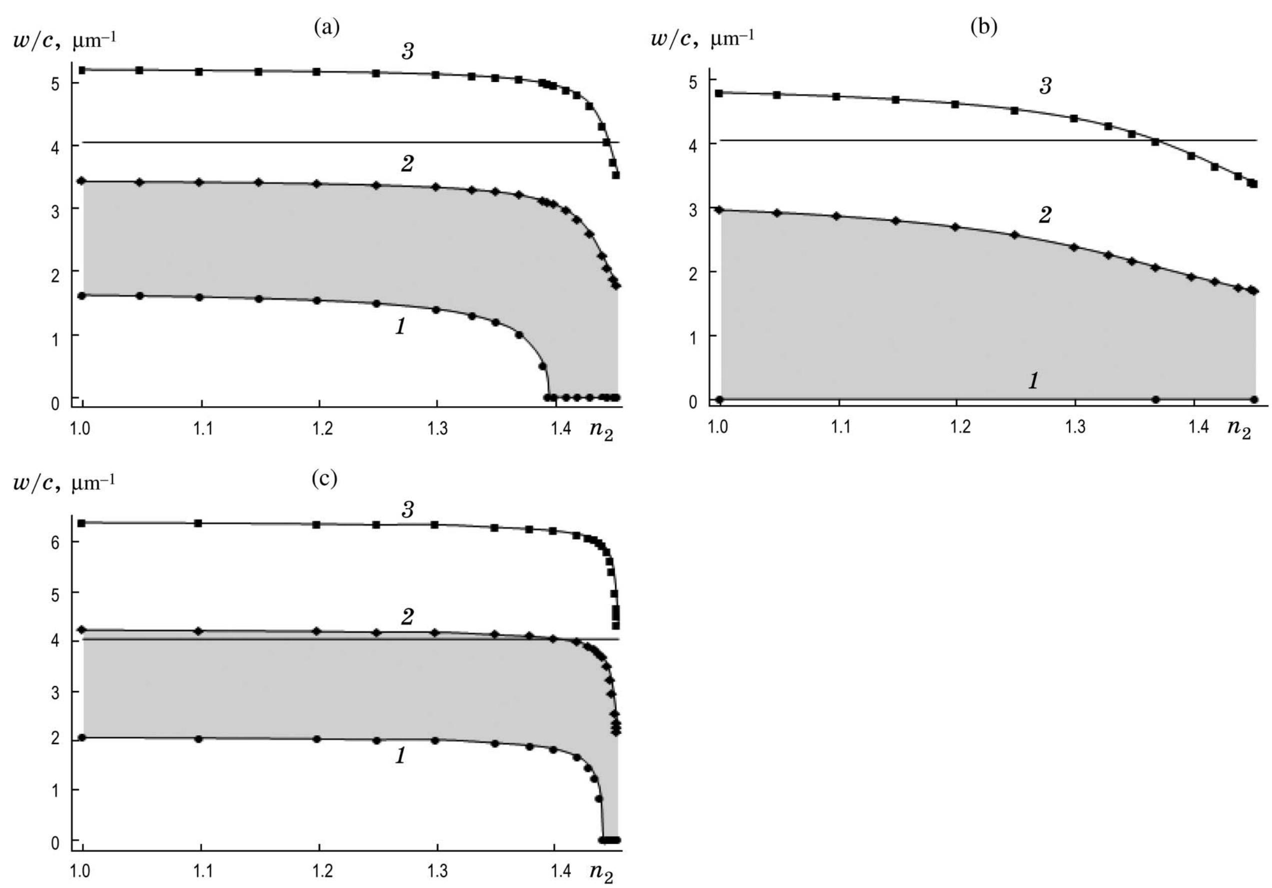

FIG. 2. Cutoff frequencies of the fundamental mode (*1*), the first odd mode (*2*), and the second even (*3*) mode of a W-lightguide with parameters $n_1 = 1.456$, $n_3 = 1.453$, $a = 9.5$ μm, and $b = 10$ μm (a); $n_1 = 1.456$, $n_3 = 1.453$, $a = 9.93$ μm, and $b = 10$ μm (b); $n_1 = 1.456$, $n_3 = 1.454$, $a = 9.5$ μm, and $b = 11$ μm (c) versus the refractive index n_2 of the intermediate layer ($1 \leq n_2 \leq n_3$).

2 μm ($\omega/c \approx 3.14\ \mu m^{-1}$), it becomes bimodal in the entire interval $1 \leq n_2 \leq n_3$.

It is easy to choose the parameters of a waveguide structure so that it is single-mode for radiation at a definite wavelength—for example, 1.55 μm. It follows from the calculation given in Fig. 2(c) for the zones of the single-mode and two-mode regimes of a W-lightguide with parameters $n_1 = 1.456$, $n_3 = 1.454$, $a = 9.5$ μm, and $b = 11$ μm that, at a wavelength of 1.55 μm, the given W-lightguide is single-mode in the range $1 < n_2 < 1.4$, whereas it becomes two-mode when $1.4 < n_2 < n_3$.

For definiteness, let us consider a W-lightguide with a reduced intermediate layer ($n_1 = 1.456$, $n_3 = 1.453$, $a = 9.93$ μm, and $b = 10$ μm). Such a structure is obviously not optimal but is convenient for representing the results of a calculation of the mode characteristics in the entire range of variation of the refractive index n_2 of the intermediate layer ($1 \leq n_2 \leq n_3$). The form of the resulting dependences will have a common character for any planar W-lightguides.

Most importantly, we used the dispersion Eqs. (6a) and (6b) to calculate the transverse wave numbers u, w, and v, which determine the transverse field distribution given by Eq. (4) for the directed (undamped) modes, in the internal, intermediate, and external layers of the waveguide, respectively, as well as the propagation constant β given by Eq. (5). Figure 3 shows how the external transverse number v [$\lambda = 1.55$ (curve *1*), 2 μm (curve *2*)] and the internal transverse number u [$\lambda = 1.55$ (curve *3*), 2 μm (curve *4*)] of the fundamental mode, as well as the external v [$\lambda = 1.55$ (curve *5*)] and internal u [$\lambda = 1.55$ (curve *6*)] transverse wave numbers of the first odd mode depend on parameter n_2. Both v and u vary insignificantly in the entire interval $1 \leq n_2 \leq n_3$; moreover, the internal wave number u weakly depends on the wavelength by comparison with the external wave number v. However, such a dependence of the wave numbers is characteristic of the modes far from their cutoff. Close to it, this dependence shows up very distinctly. Thus, a second even mode appears at $n_2 = 1.369$, at a wavelength of 1.55 μm. The external and

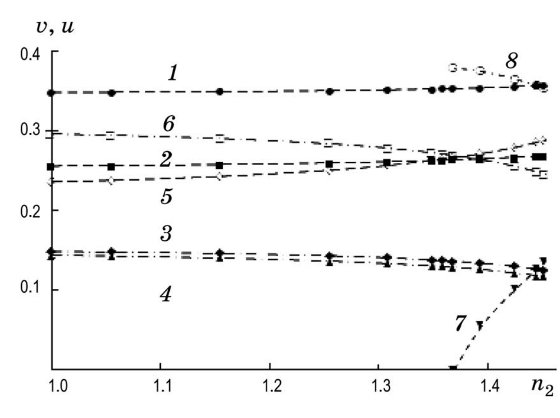

FIG. 3. External transverse wave number v of the fundamental mode [$\lambda = 1.55$ (*1*), 2 μm (*2*)] and its internal transverse wave number u [$\lambda = 1.55$ (*3*), 2 μm (*4*)], as well as the external and internal wave numbers of the first odd mode [$\lambda = 1.55$ (*5*) and (*6*)] and the second even mode [$\lambda = 1.55$ (*7*) and (*8*)] of a W-lightguide with parameters $n_1 = 1.456$, $n_3 = 1.453$, $a = 9.93$ μm, and $b = 10$ μm versus the refractive index n_2 of the intermediate layer.

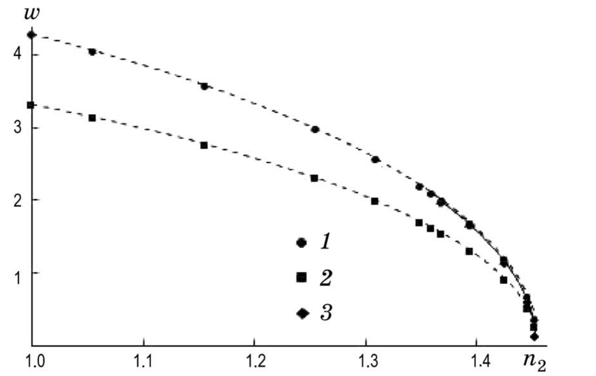

FIG. 4. Intermediate transverse wave number w of the fundamental mode [$\lambda = 1.55$ (*1*), 2 μm (*2*)], as well as intermediate wave number of the second even mode [$\lambda = 1.55$ (*3*)] of a W-lightguide with parameters $n_1 = 1.456$, $n_3 = 1.453$, $a = 9.93$ μm, and $b = 10$ μm versus the refractive index n_2 of the intermediate layer.

internal wave numbers of the second even mode in the interval $1.369 \leq n_2 \leq n_3$ now form curves *7* and *8*, respectively, demonstrating a steeper variation rate.

The result of calculating the intermediate transverse wave number w of the fundamental mode [$\lambda = 1.55$ (curve *1*) and 2 μm (curve *2*)] and the second even mode [$\lambda = 1.55$ μm (curve *3*)], shown in Fig. 4, indicates that w strongly (by comparison with v and u—see Fig. 3) depends on the refractive index n_2 of the intermediate layer and the wavelength of the radiation. The dependence of w on n_2 for the first odd mode, at wavelength 1.55 μm, virtually coincides with the similar dependence for the fundamental mode $(w_{\text{even}} - w_{\text{odd}})/w \approx 0.01$ and therefore is not shown here.

As can be seen from Fig. 5, reducing n_2 (strengthening the contrast of the waveguide's refractive-index profile) increases the field concentration in the light-conducting layer, and this in turn substantially reduces the losses at a bend in W-lightguides by comparison with standard stepped waveguides. Moreover, the field concentration of the fundamental mode at a wavelength of 1.55 μm (curve *1*) exceeds the analogous one at $\lambda = 2$ μm (curve *2*). The behavior of the dependence of the field concentration of the first odd mode at $\lambda = 1.55$ μm has a similar character (curve *3*). The degree of the field

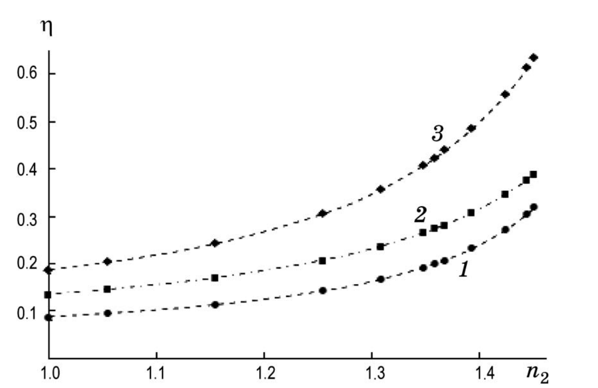

FIG. 5. Field concentration of the fundamental mode [$\lambda = 1.55$ (*1*), 2 μm (*2*)] and the first odd mode [$\lambda = 1.55$ (*3*)] of the W-lightguide considered here versus the refractive index n_2 of the intermediate layer.

concentration was estimated by coefficient $\eta = E(b)/E_{max}$ (a smaller value of the coefficient corresponds to a larger field concentration)[2]

$$\eta = \left| \frac{w \cos(ua)}{w\,\mathrm{ch}[w(b-a)] + v\,\mathrm{ch}[w(b-a)]} \right| \quad \text{for even modes,} \tag{16a}$$

$$\eta = \left| \frac{w \sin(ua)}{w\,\mathrm{ch}[w(b-a)] + v\,\mathrm{ch}[w(b-a)]} \right| \quad \text{for odd modes.} \tag{16b}$$

Evanescent modes play an appreciable role in a W-lightguide. Such modes, as pointed out above, correspond to the complex roots of dispersion Eqs. (6a) and (6b) of the external transverse wave numbers $v = v' + iv''$, which determine the imaginary part of the propagation constant

$$\beta = \beta' - i\beta'' = \sqrt{k^2 n_3^2 + v^2} \tag{17}$$

and, in the final analysis, the losses to radiation given by Eq. (12). The distribution of roots on the complex plane has the lobed structure shown in Fig. 6(a),[2] with the minimum losses possessed by the evanescent modes with the smallest imaginary part $|\beta''|$. Therefore, to stabilize the directed-mode regime, the W-lightguide parameters need to be chosen to ensure effective filtering of the evanescent (parasitic) modes, which possess the minimum losses. To do this, it is useful to track how the wave numbers v of the indicated modes behave as the refractive index of the intermediate layer varies. The first fourteen values of the external transverse wave number v (curves *1* and *2*) for the evanescent modes at $\lambda = 1.55$ μm and the same number of values of v (curves *3* and *4*) at $\lambda = 2$ μm, computed for $n_2 = 1$, are shown in Fig. 6(b). The roots given by curves *1* and *3* are solutions of dispersion Eq. (6a) for even modes, while the roots given by curves *2* and *4* are the solutions of dispersion Eq. (6b) for odd modes. The roots that correspond to the minimum losses (the first three for the evanescent modes at wavelength 1.55 μm and one for the evanescent mode at wavelength 2 μm) are shown by an ellipse. Their dependences on the refractive index of the intermediate layer are shown in Fig. 7. It can be seen from the figure that, as n_2 increases (as the contrast of the refractive-index profile decreases), the imaginary values $|v''|$ of the roots decrease. Moreover, for a definite value of the refractive index (in this case, when $n_2 \approx 1.36$), the imaginary part of the wave number of the evanescent mode with minimal losses at $\lambda = 1.55$ μm becomes equal to zero ($v = -0.0625$, point *7*). The position of point *7* is found by solving the following system of equations:

$$\begin{cases} F(v, n_2) = 0 \\ F'(v, n_2) = 0 \end{cases}. \tag{18}$$

A further increase of the refractive index of the intermediate layer by $\Delta n_2 \approx 0.01$ causes a second even mode to appear at wavelength 1.55 μm with $n_2 \approx 1.37$ and $v = 0$ (see Fig. 3, curves *7* and *8*).

It follows from what was said above that, in a W-lightguide with a fairly contrast refractive-index profile, it is possible to ensure a high degree of concentration of the directed mode and effective filtering of the evanescent modes in the given wavelength region. For instance, for a structure with $n_1 = 1.456$, $n_3 = 1.454$, $a = 9.5$ μm, and $b = 11$ μm in which $n_2 = 1.38$, the field concentration in the fundamental mode at a wavelength of 1.55 μm is $\eta \approx 0.009$, while the damping of the evanescent mode with the smallest losses is $\alpha \approx 54$ dB/m.

In this case, the refractive index n_2 of the intermediate layer must not exceed the critical value n_{2k}, for which the losses of the evanescent mode at the working frequency become negligible.

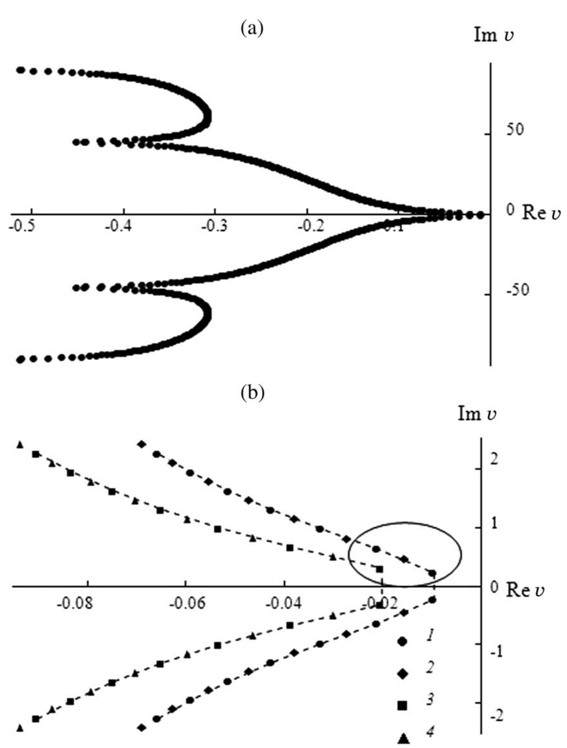

FIG. 6. Calculated values of the external transverse wave number v of the evanescent modes ($\lambda = 1.55$ μm), computed in the range $|v''| < 90$ (a) and of the fist fourteen (the smallest in modulus) values of v [$\lambda = 1.55$ (*1*) and (*2*), 2 μm (*3*) and (*4*)] (b) of the W-lightguide considered here when $n_2 = 1$.

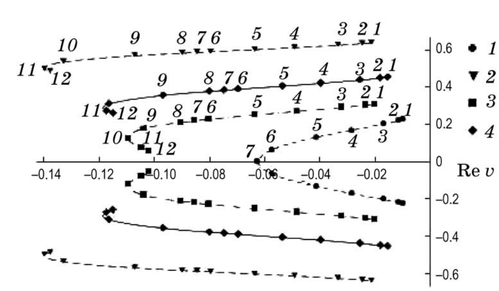

FIG. 7. Dependence of the three (smallest in modulus) roots of the dispersion equation [$\lambda = 1.55$ (*1*, *2*, *4*)] and of the one (smallest in modulus) root [$\lambda = 2$ μm (*3*)] (circled in Fig. 6) for various values of the refractive index of the intermediate layer $n_2 = 1$ (*1*), 1.056 (*2*), 1.156 (*3*), 1.256 (*4*), 1.31 (*5*), 1.35 (*6*), 1.36 (*7*), 1.37 (*8*), 1.39 (*9*), 1.426 (*10*), 1.446 (*11*), and 1.453 (*12*).

We should point out that the field distribution in the external field of the waveguide sharply varies when n_2 exceeds the critical value n_{2k} but remains smaller than the value at which a new mode appears: $n_{2k} \leq n_2 \leq n_{2k} + \Delta n_2$ (in the case considered here, $\Delta n_2 \approx 0.01$), and this can be used in developing waveguide splitters and electro-optic modulators.

The size of the Δn_2 interval depends both on the radiation frequency and on the geometrical and optical parameters of the W-lightguide. For instance, in the zone of the two-mode regime in Fig. 2(a), it has the value $\Delta n_2 \approx 0.001$ at wavelength 1.55 μm.

CONCLUSION

The planar five-layer model has been used to obtain and analyze the relationship of the modal dispersion of a W-lightguide in a wide range of variation of its parameters. In particular, the behavior of the directed and evanescent modes as the contrast of the refractive-index profile varies has been investigated in fairly great detail.

It has been shown that a special role is played by the width and refractive index of the intermediate layer in a wide range of parameters that determine the properties of a W-lightguide. The choice of suitable values of just these parameters can substantially expand the zone of the single-mode regime, can optimize the size of the fundamental mode, can reach a high field concentration, and, as a consequence, can obtain small bending losses of the radiation.

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