ISSN 1064-2269, Journal of Communications Technology and Electronics, 2015, Vol. 60, No. 5, pp. 480–488. © Pleiades Publishing, Inc., 2015.
Original Russian Text © A.P. Anyutin, I.P. Korshunov, A.D. Shatrov, 2015, published in Radiotekhnika i Elektronika, 2015, Vol. 60, No. 5, pp. 513–521.

ELECTRODYNAMICS AND WAVE PROPAGATION

Coupled Quasi-Static Oscillations in Two Cylinders Made of a Metamaterial

A. P. Anyutin, I. P. Korshunov, and A. D. Shatrov

Kotel'nikov Institute of Radio Engineering and Electronics, Russian Academy of Sciences (Fryazino Branch), pl. Vvedenskogo 1, Fryazino, Moscow oblast, 141190 Russia

e-mail: anioutine@mail.ru; korip@ms.ire.rssi.ru

Received September 24, 2014

Abstract—A two-dimensional problem of excitation, by a linear source, of a structure consisting of two identical solid or hollow cylinders made of a metamaterial under the condition that the geometric and material parameters of the cylinders provide for the existence of high-Q quasi-static resonances in each of the cylinders is considered. The effect of the distance between the cylinders on the spatial structure of the resonant fields is studied. It is shown that the frequency response of the structure has the shape characteristic of the response of coupled resonators. The behavior of the fields in the vicinity of the critical coupling, which qualitatively changes the frequency characteristics of the composite resonator, is numerically investigated.

DOI: 10.1134/S1064226915040026

INTRODUCTION

A two-dimensional problem of excitation, by a filamentary magnetic current, of two identical hollow cylinders made of a material whose permittivity ε and permeability μ are negative is considered (Fig. 1). The following designations are used: a is the radius of the inner cavities, b is the outer radius of the cylinders, and d is the distance between the centers of the cylinders. The special case of $a = 0$ corresponds to solid cylinders. Two polar coordinate systems are used: (r_1, φ_1) and (r_2, φ_2). They are associated with the centers of the left and right cylinders, respectively. The case of the TM polarization is under study, for which the two-dimensional wave field $U = U_1(r_1, \varphi_1) = U_2(r_2, \varphi_2)$ has the sense of the z component of the magnetic field: $U = H_z$.

The source of the cylindrical wave is located on the straight line connecting the centers of the cylinders. In

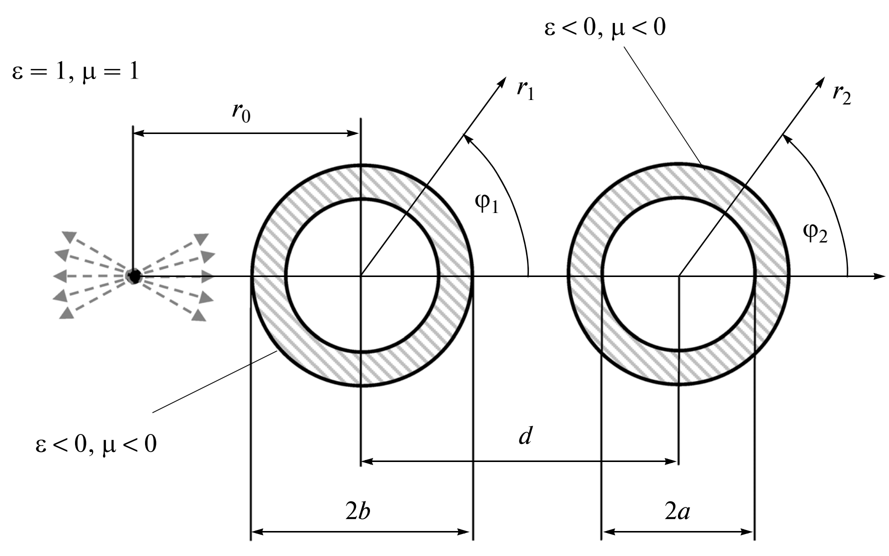

Fig. 1. Geometry of the problem.

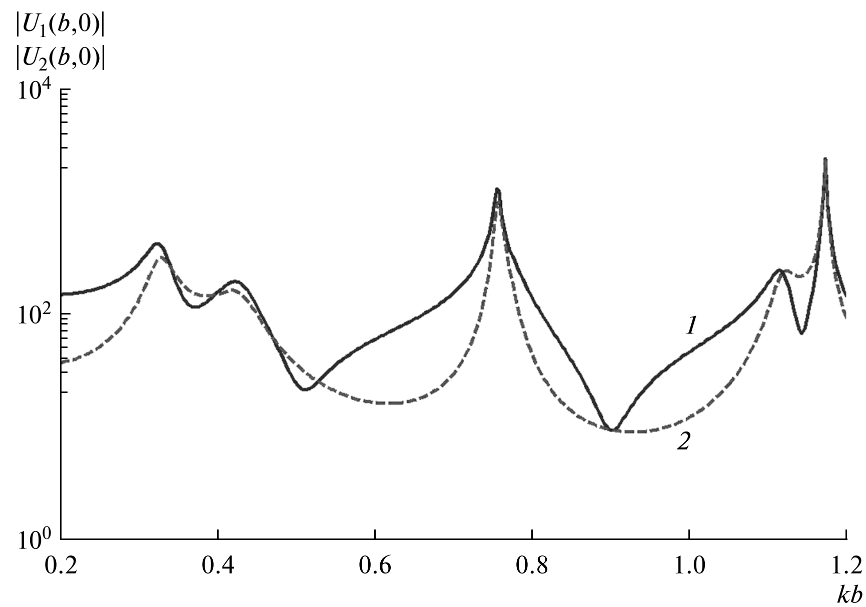

Fig. 2. Amplitude–frequency characteristics of a structure consisting of two solid cylinders at $d = 2.1b$: (*1*) $|U_1(b,0)|$ and (*2*) $|U_2(b,0)|$.

variables (r_1, φ_1) and (r_2, φ_2), the exciting field is defined as

$$U^0 = H_0^{(2)}\left(k\sqrt{r_1^2 + r_0^2 + 2r_1r_0\cos\varphi_1}\right)$$
$$= H_0^{(2)}\left(k\sqrt{r_2^2 + (r_0 + d)^2 + 2r_2(r_0 + d)\cos\varphi_2}\right), \quad (1)$$

where $H_0^{(2)}$ is the Hankel function and k is the free-space wave number.

The problem of excitation of solid and hollow single cylinders made of a metamaterial is considered in studies [1, 2]. It has been found that, at the values of ε close to minus unity, high-Q resonances exist in the cylinders of small electrical radius $(kb \ll 1)$. Under the resonance condition, the wave field is described by single azimuthal harmonic $\cos(m\varphi)$. In this case, the field is localized near the cylinder surfaces $r = b$ and $r = a$. These cylinders can be considered as ring resonators based on the use of strongly slowed surface waves propagating along the boundary of the metamaterial.

The objective of the study is to investigate the resonance phenomena in the structure containing two identical cylinders. The interaction of their resonance fields results in the structure's frequency response whose shape is characteristic of a system of two coupled resonators. The parameter that determines the coupling between the resonators is distance d between the cylinders. The region of values of d that corresponds to the critical coupling at which the structure's frequency characteristic has a dip has been determined.

Among the studies that are close to the research area under consideration, the investigations of coupled plasmon oscillations of two spherical nanoparticles can be mentioned (e.g., see monograph [3] and the literature therein).

1. NUMERICAL RESULTS

The numerical calculations of the wave fields have been performed with the use of the modified discrete-source method [4], which is used in the investigations of the resonance properties of single cylinders made of metamaterials in [1, 2].

Let us present the results obtained for solid cylinders $(a = 0)$. In all calculations, it is assumed that $r_0 = 1.2b$. The metamaterial parameters are the following:

$$\varepsilon = -1.01, \quad \mu = -0.91. \quad (2)$$

The frequency response of the structure will be characterized by the values of the magnitude of the wave field at points $r_1 = b$, $\varphi_1 = 0$ and $r_2 = b$, $\varphi_2 = 0$, which are located at the boundaries of the cylinders. Figure 2 shows the amplitude–frequency characteristics (AFCs) of the structure for a small distance between the cylinders $(d = 2.1b)$. There are several resonance peaks on curves *1* and *2*. The lowest resonance frequency is $kb = 0.33\ldots$. Figure 3 shows the AFC of a single cylinder $(d = \infty)$ with the above-mentioned material parameters. A comparison of the curves given in Figs. 2 and 3 indicates that the resonance frequencies of the two-element structure significantly differ from the resonance frequencies of a single cylinder.

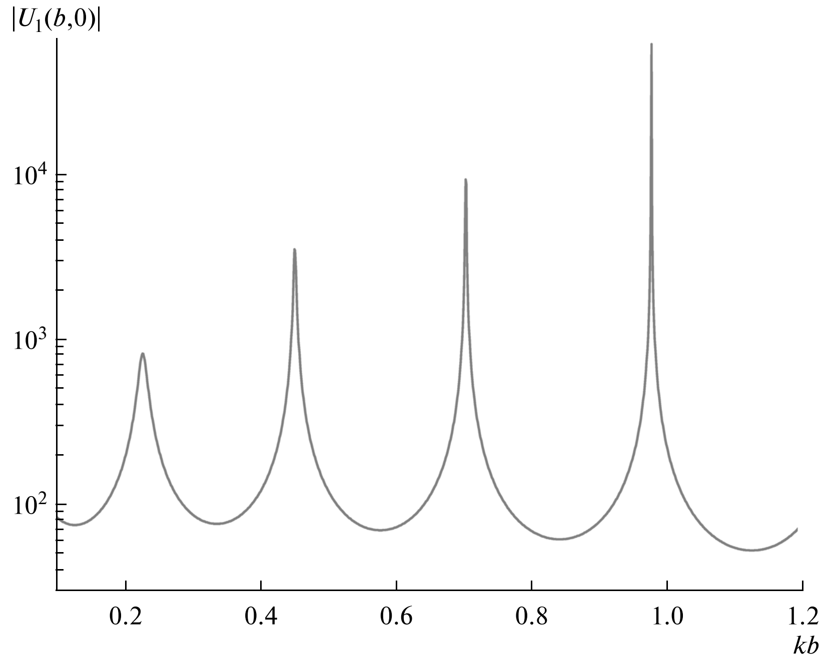

Fig. 3. Amplitude–frequency characteristic of a single solid cylinder.

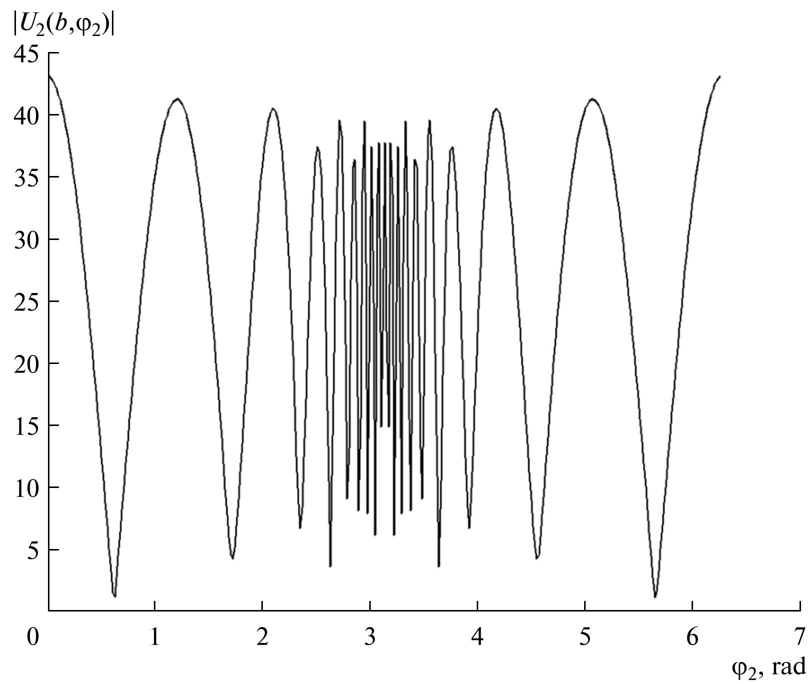

Fig. 4. Distribution of the magnitude of field $|U_2(b,\varphi_2)|$ along the boundary of the solid left cylinder at the resonance frequency $kb = 0.33\ldots$ for $d = 2.1b$.

Figure 4 shows the distribution of the magnitude of the field $U_2(b,\varphi_2)$ along the boundary of the right cylinder at the resonance frequency $kb = 0.33\ldots$. The calculations show that the resonance fields on the left and right cylinders satisfy the relation $|U_1(b,\varphi)| = |U_2(b,\pi-\varphi)|$ to a high degree of accuracy. An unusual property of these complex distributions is the presence of ultrafast oscillations in the regions $\varphi_1 \simeq 0$ and $\varphi_2 \simeq \pi$. These regions correspond to the points of the closest approach of the boundaries of the cylinders.

Let us now show that this behavior of the field is due to the presence of two closely spaced interfaces

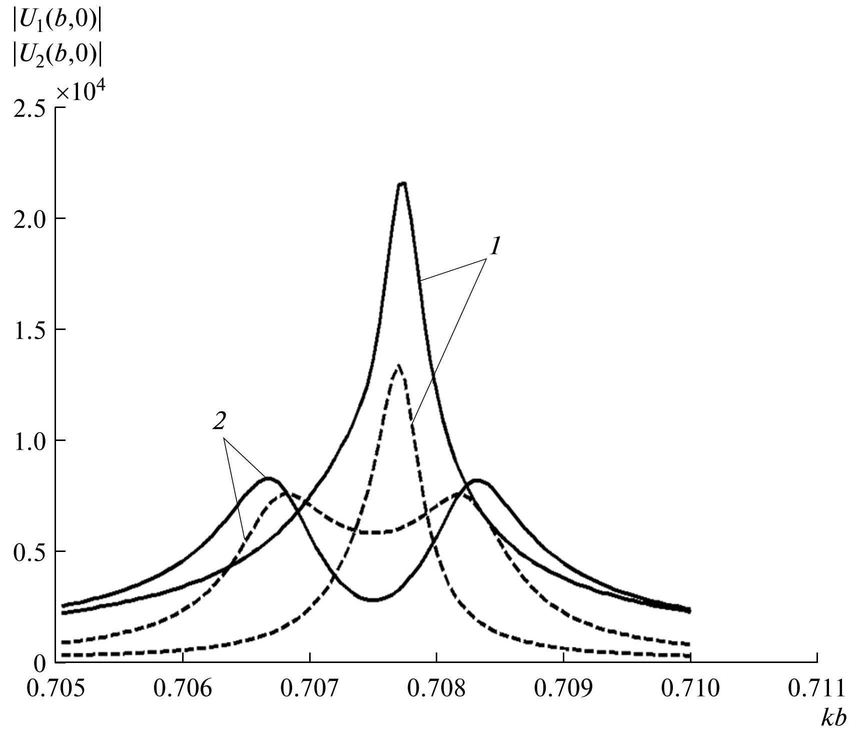

Fig. 5. Amplitude–frequency characteristics of a structure consisting of two solid cylinders: (solid lines) $|U_1(b,0)|$ and (dashed lines) $|U_2(b,0)|$; $d = (1)\ 10b$ and $(2)\ 7.7b$.

between the metamaterial and free space. It is physically obvious that, in the region of the closest approach of the cylinders ($r_1 \approx b$, $\varphi_1 \approx 0$), the geometry of a composite body can be presented in a simplified form as a plane-parallel gap of the thickness $\delta = d - 2b$ that separates two half-spaces with material parameters ε and μ. At the values of ε and μ determined by formula (2), in the small-thickness gap, there exists an even surface mode whose propagation constant h satisfies the condition $h \gg k$ [5]. The curve in Fig. 4 has been calculated for the parameters $kb = 0.33\ldots$ and $d = 2.1b$. This means that the electrical thickness of the gap is $k\delta = 0.033\ldots$ With a knowledge of quantities $k\delta$, ε, and μ, from the corresponding dispersion relations (see [5]), we obtain $h/k \approx 160$. As a result, the field of the standing surface wave in the region $r_1 = b$, $\varphi_1 \approx 0$ is described by the function $\cos(hb\varphi_1) \approx \cos(53\varphi_1)$. The oscillation period of the field $U_2(b,\varphi_2)$ near the direction $\varphi_2 = \pi$ (see Fig. 4) agrees with the estimate obtained from the above-mentioned formula. Note that, in the case of a single cylinder, the resonance field is described by a single azimuthal harmonic $\cos(m\varphi)$.

Thus, from the above-mentioned results, it follows that a structure consisting of two closely spaced cylinders, as well as a single-cylinder structure, exhibits resonant properties in the lower-frequency region. However, spatial frequency characteristics of the resonant fields differ fundamentally from the corresponding characteristics of the fields of a single cylinder.

It is obvious that, if the distance between the cylinders is sufficiently large, the interaction of the partial quasi-static fields (each field is concentrated in the neighborhood of the corresponding cylinder) decreases. In this case, near-cylinder fields $U_1(r_1,\varphi_1)$ and $U_2(r_2,\varphi_2)$ have the same spatial structure and differ only in amplitudes. As a result, the structure AFC $|U_1(b,0)|$ has the same shape as the AFC of a single cylinder.

It is of interest to investigate the intermediate region of distances d in which the interaction between the cylinders results in a qualitative change in the AFC. Figure 5 shows the AFCs for $d = 10b$ and $d = 7.7b$ (curves *1* and *2*, respectively). The computational results are presented for a narrow frequency band corresponding to the forth-harmonic resonance. In this frequency band, the field at the boundaries of the cylinders are very accurately described by functions $\cos(4\varphi_{1,2})$. This is illustrated by Fig. 6, which shows functions $U_1(b,\varphi_1)$ and $U_2(b,\varphi_2)$ for $d = 7.7b$. As distance d increases, the accuracy of the approximate description must improve, which is confirmed by the numerical calculations. It follows from Fig. 5 that at $d = 10b$ the resonance curves have a single peak, whereas at $d = 7.7b$ a dip is formed on the resonance curves.

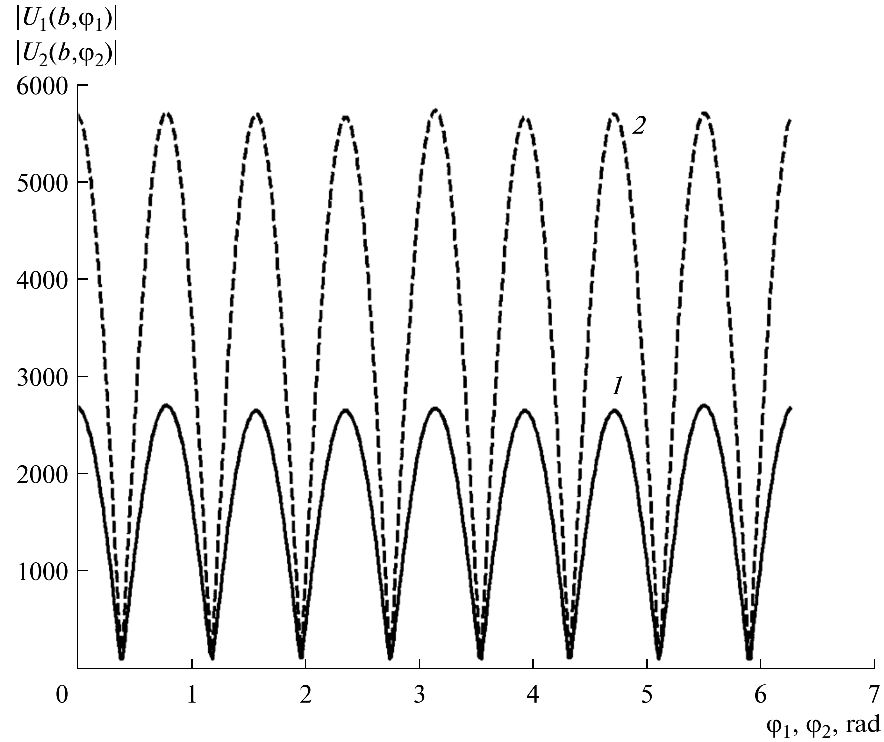

Fig. 6. Distribution of field magnitudes (*1*) $|U_1(b,\varphi_1)|$ and (*2*) $|U_2(b,\varphi_2)|$ along the boundaries of solid cylinders at the resonance frequency $kb = 0.7075$ at $d = 7.7b$.

Figure 7 illustrates the behavior of the field magnitude along the line connecting the centers of the cylinders at the coupling between the partial resonators that is below the critical level ($d = 10b$, curve *1*) and at the coupling exceeding the critical value ($d = 7.7b$, curve *2*). The calculations were performed for the central frequencies of the corresponding resonance curves. It can be seen that at $d = 10b$ the amplitude of the field at the left cylinder is higher than at the right one, whereas at $d = 7.7b$ the ratio between the amplitudes is reversed. Note that the level of the field at the boundaries of the cylinders is much higher than the level of the field at the gap midpoint $r_1 = d/2$.

Figure 8 shows the magnitude of the scattering pattern $\left|\Phi^{S}(\varphi_1)\right|$ for two cylinders spaced by the distance $d = 7.7b$. The scattering pattern is determined by the formula

$$U_1^{S}(r_1,\varphi_1) = \Phi^{S}(\varphi_1)(2/\pi k r_1)^{1/2}\exp(-ikr_1 + i\pi/4),\quad kr_1 \to \infty, \tag{3}$$

where $U_1^{S}(r_1,\varphi_1)$ is the scattered field in coordinate system (r_1,φ_1).

It should be noted that the scattering pattern of a composite object with a total length of the order of a wavelength is a complex lobed one, with the lobe amplitudes considerably exceeding the level of the primary-field pattern $\left|\Phi_0\right| = 1$. These “superdirectivity” properties are characteristic of the scattering by high-Q resonant objects [6].

2. APPROXIMATE ANALYTICAL DESCRIPTION OF COUPLED RESONANCES

From the numerical results given in the previous section, it follows that, if the distance between the cylinders is sufficiently large, the structure of the resonant wave field in the vicinity of each cylinder is close to the structure of the field considered in the problem of excitation of a single cylinder. This circumstance allows us to use the results obtained in studies [1, 2] in the approximate investigation of the problem under discussion.

The problem of diffraction of the field

$$U_m^0 = J_m(kr)\cos(m\varphi) \tag{4}$$

by a single cylinder has a simple solution [1, 2]. Outside the cylinder $(r > b)$, a scattered field arises, which is determined by the formula

$$U_m^{S} = R_m H_m^{(2)}(kr)\cos(m\varphi). \tag{5}$$

Coefficient of “reflection” R_m for a solid cylinder $(a = 0)$ is

$$R_m = -\frac{A_m(kb)}{C_m(kb)}, \tag{6}$$

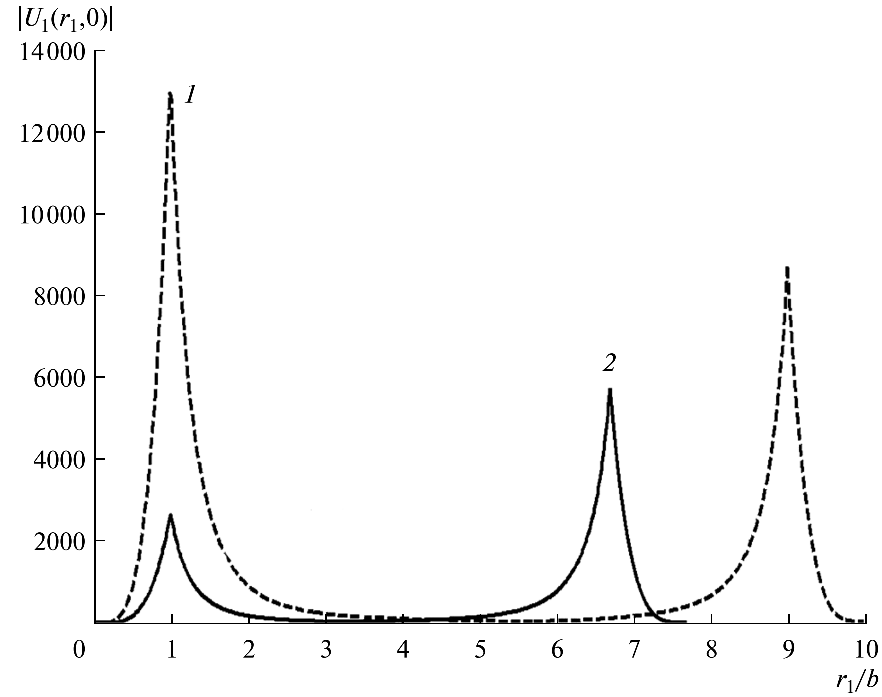

Fig. 7. Distribution of field magnitude $|U_1(r_1, 0)|$ along the line connecting the centers of the solid cylinders at the resonance frequency $kb = 0.7075$: $d = (1)\ 10b$ and $(2)\ 7.7b$.

where

$$A_m(kb) = J'_m(kb)J_m(Nkb) - \frac{N}{\varepsilon}J_m(kb)J'_m(Nkb), \quad (7)$$

$$C_m(kb) = H_m^{(2)}(kb)J_m(Nkb) - \frac{N}{\varepsilon}H_m^{(2)}(kb)J'_m(Nkb), \quad (8)$$

$$N = \sqrt{\varepsilon\mu}, \quad (9)$$

and J_m are the Bessel functions, the prime indicating differentiation with respect to the argument.

For a hollow cylinder, we have

$$R_m = -\frac{A_m(ka)B_m(kb) - B_m(ka)A_m(kb)}{A_m(ka)D_m(kb) - B_m(ka)C_m(kb)}, \quad (10)$$

where

$$B_m(kb) = J'_m(kb)H_m^{(2)}(Nkb) - \frac{N}{\varepsilon}J_m(kb)H_m^{(2)'}(Nkb), \quad (11)$$

$$D_m(kb) = H_m^{(2)'}(kb)H_m^{(2)}(Nkb) - \frac{N}{\varepsilon}H_m^{(2)}(kb)H_m^{(2)'}(Nkb). \quad (12)$$

The high-Q quasi-static resonances occur at the frequency determined by the equation

$$R_m = -1. \quad (13)$$

It follows from formulas (4), (5), and (13) that, at $kb \ll 1$, the amplitude of the scattered field at the boundary of the cylinder is much higher than the amplitude of the incident field:

$$\left|\frac{U_m^S}{U_m^0}\right| = \left|\frac{H_m^{(2)}(kb)}{J_m(kb)}\right| \gg 1. \quad (14)$$

Therefore, it can be assumed that, in a system of two cylinders that are at a sufficient distance from each

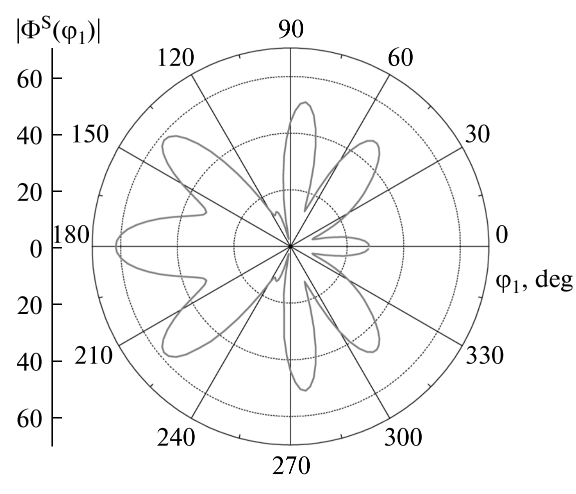

Fig. 8. Magnitude of the scattering pattern of a structure consisting of two solid cylinders $|\Phi^S(\varphi_1)|$ at the resonance frequency $kb = 0.7075$ at $d = 7.7b$.

other, under the condition of the mth harmonic resonance, the total field outside the cylinders consists of the following three terms:

$$U = U^0 + \alpha_m H_m^{(2)}(kr_1)\cos(m\varphi_1) + \beta_m H_m^{(2)}(kr_2)\cos(m\varphi_2). \quad (15)$$

Unknown amplitudes α_m and β_m can be found from the system of algebraic equations

$$\alpha_m = (-1)^m R_m \left\{2H_m^{(2)}(kr_0) + \beta_m\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]\right\}, \quad (16)$$

$$\beta_m = (-1)^m R_m \left\{2H_m^{(2)}(kr_0 + kd) + \alpha_m\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]\right\}. \quad (17)$$

This system follows from relations (4), (5) and the summation theorems for cylindrical functions [7]

$$H_m^{(2)}(kr_1)\cos(m\varphi_1) = \sum_{l=-\infty}^{\infty} H_{m-l}^{(2)}(kd)J_l(kr_2)\cos(l\varphi_2), \quad (18)$$

$$H_m^{(2)}(kr_2)\cos(m\varphi_2) = \sum_{l=-\infty}^{\infty} H_{l-m}^{(2)}(kd)J_l(kr_1)\cos(l\varphi_1), \quad (19)$$

$$H_0^{(2)}\left[k\sqrt{r_1^2 + r_0^2 + 2r_1r_0\cos\varphi_1}\right] = \sum_{l=-\infty}^{\infty}(-1)^l H_l^{(2)}(kr_0)J_l(kr_1)\cos(l\varphi_1), \quad (20)$$

$$H_0^{(2)}\left[k\sqrt{r_2^2 + (r_0+d)^2 + 2r_2(r_0+d)\cos\varphi_2}\right] = \sum_{l=-\infty}^{\infty}(-1)^l H_l^{(2)}(kr_0 + kd)J_l(kr_2)\cos(l\varphi_2). \quad (21)$$

The solution to system (16), (17) has the form

$$\alpha_m = 2R_m \frac{(-1)^m H_m^{(2)}(kr_0) + R_m H_m^{(2)}(kr_0 + kd)\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]}{1 - R_m^2\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]^2}, \quad (22)$$

$$\beta_m = 2R_m \frac{(-1)^m H_m^{(2)}(kr_0 + kd) + R_m H_m^{(2)}(kr_0)\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]}{1 - R_m^2\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]^2}. \quad (23)$$

Let us present the results of the investigations of the structure consisting of two hollow cylinders performed by the above-described method. Figure 9 shows frequency dependences $|U_1(b,0)|$ and $|U_2(b,0)|$ calculated with the use of formula (15) for two distances between the cylinders: $d = 7.8b$ and $d = 5.4b$. The general view of the curves in Fig. 9 is identical to that of the AFCs for solid cylinders (see Fig. 5). Specifically, when the cylinders are sufficiently close to each other, two peaks separated by a dip appear on the resonance curve.

Let us show that the splitting of the resonance curves is related to the existence of symmetric and antisymmetric oscillations in a structure consisting of two identical cylinders. Let us consider the case when the cylinders are excited by two sources and the incident field has the form (Fig. 10)

$$U_{\pm}^0 = H_0^{(2)}\left[k\sqrt{r_1^2 + r_0^2 + 2r_1r_0\cos\varphi_1}\right] \pm H_0^{(2)}\left[k\sqrt{r_2^2 + r_0^2 - 2r_2r_0\cos\varphi_2}\right]. \quad (24)$$

It is obvious that in this case the total field, as well as the incident field, will be symmetric (plus sign) or antisymmetric (minus sign) with respect to the midplane $r_1 = d/2\cos\varphi_1$ separating the cylinders. Under the condition of the mth harmonic resonance, the field outside the cylinders will be determined by formula (15), in which coefficients α_m and β_m must be presented in the form

$$\alpha_m^+ = 2R_m \frac{(-1)^m H_m^{(2)}(kr_0) + H_m^{(2)}(kr_0 + kd)}{1 - R_m\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]}, \quad (25)$$

$$\beta_m^+ = (-1)^m \alpha_m^+, \quad (26)$$

$$\alpha_m^- = 2R_m \frac{(-1)^m H_m^{(2)}(kr_0) - H_m^{(2)}(kr_0 + kd)}{1 + R_m\left[(-1)^m H_0^{(2)}(kd) + H_{2m}^{(2)}(kd)\right]}, \quad (27)$$

$$\beta_m^- = -(-1)^m \alpha_m^-. \quad (28)$$

It is readily seen that primary cylindrical wave U^0 (see (1)) excites the field that is a half-sum of the fields arising at the symmetric $\left(U_+^0\right)$ and antisymmetric $\left(U_-^0\right)$ excitations. Each of these summands is characterized by its own resonance frequency. Figure 11 shows a family of curves that describes coefficients α_4^+ and α_4^- as functions of frequency at various distances d between the centers of the solid cylinders. It follows from the figure that, at $d = 9b$, the resonance frequencies of the symmetric and antisymmetric oscillations are practically the same. As distance d decreases, the

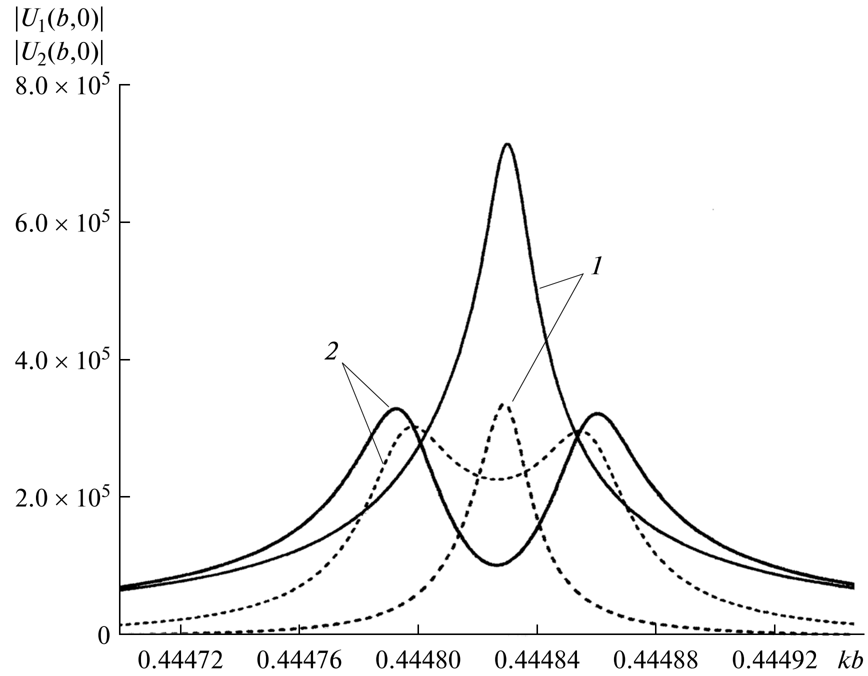

Fig. 9. Amplitude–frequency characteristics of a structure consisting of two hollow cylinders at $a = 0.25b$: (solid lines) $|U_1(b,0)|$ and (dashed lines) $|U_2(b,0)|$; $d =$ (*1*) 7.8b and (*2*) 5.4b.

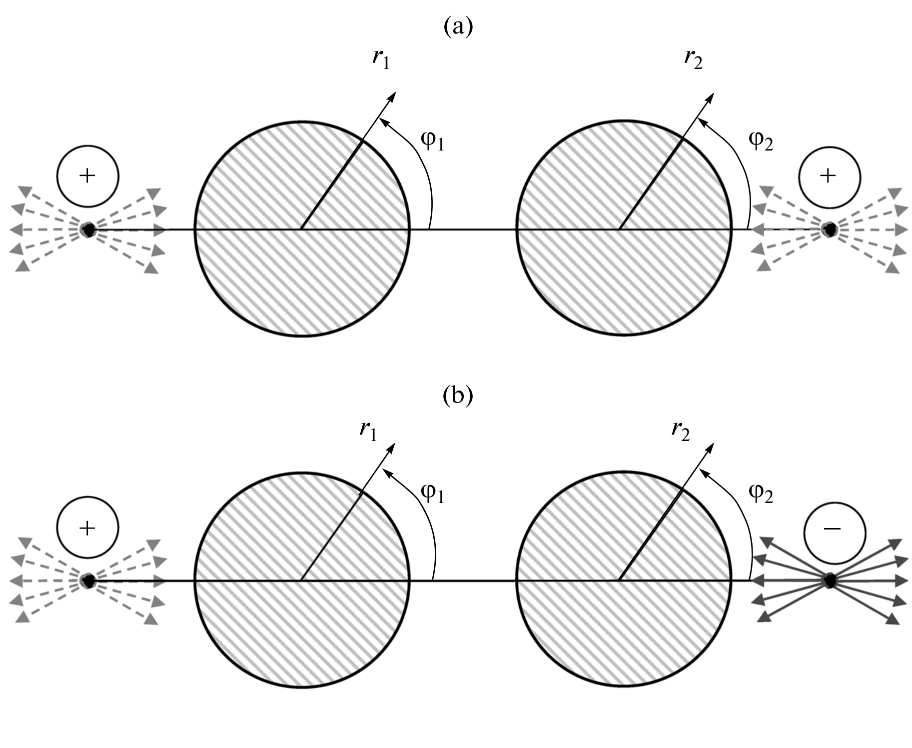

Fig. 10. (a) Symmetric and (b) antisymmetric excitation of the cylinders.

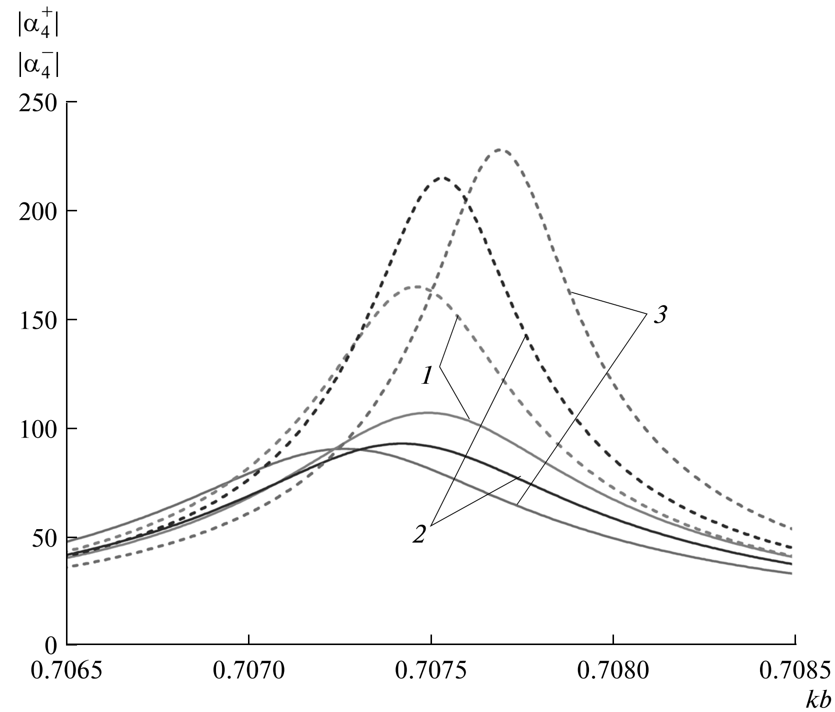

Fig. 11. Amplitudes of the fourth harmonic as functions of frequency for a symmetric and antisymmetric excitation of the structure consisting of two solid cylinders: (solid lines) $\left|\alpha_4^+\right|$ and (dashed lines) $\left|\alpha_4^-\right|$; $d =$ (*1*) $9b$, (*2*) $8b$, and (*3*) $7b$.

resonance frequencies shift in the opposite directions. When the difference between these frequencies is of the order of the width of the resonance curves, a dip appears on the frequency characteristic.

CONCLUSIONS

The coupled oscillations excited by a filamentary current in a structure consisting of two identical metamaterial cylinders have been investigated. It is shown that this structure is characterized by resonant properties in the lower-frequency region. The wave fields in structures consisting of solid and hollow cylinders have been calculated by rigorous numerical methods. The effect of the distance between the cylinders on the spectral and spatial properties of the resonance fields has been considered. It has been found that, at a certain distance between the cylinders, there occurs a splitting of the frequency characteristic of the composite resonator. It is noted that this effect can be explained by the interference between the symmetric and antisymmetric oscillations in the two-element structure.

ACKNOWLEDGMENTS

This study was supported in part by the Russian Foundation for Basic Research, project no. 15-02-00954a.

REFERENCES

1. A. P. Anyutin, I. P. Korshunov, and A. D. Shatrov, J. Commun. Technol. Electron. **58**, 926 (2013).
2. A. P. Anyutin, I. P. Korshunov, and A. D. Shatrov, J. Commun. Technol. Electron. **59**, 1087 (2014).
3. V. V. Klimov, *Nanoplasmonics* (Fizmatlit, Moscow, 2009) [in Russian].
4. A. G. Kyurkchan and N. I. Smirnova, *Mathematical Modeling in Diffraction Theory with the Use of A Priori Information on the Analytic Properties of the Solution* (ID Media Publisher, Moscow, 2014) [in Russian].
5. A. D. Shatrov, J. Commun. Technol. Electron. **52**, 842 (2007).
6. A. P. Anyutin and A. D. Shatrov, Tech. Phys. Lett. **39**, 287 (2013).
7. V. T. Erofeenko, *Addition Theorems: Handbook* (Nauka i Tekhnika, Minsk, 1989) [in Russian].

Translated by I. Nikishin