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ELECTRODYNAMICS AND WAVE PROPAGATION

Low-Frequency Resonances of a Chiral Sphere Filled with a Metamaterial

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Abstract—The problem of excitation of a metamaterial sphere by radial magnetic and electric dipoles is considered. The sphere's surface is characterized by the anisotropic conductivity along helical lines. The features of quasi-static resonances of the chiral sphere that are due to the presence of a metamaterial are investigated. It is demonstrated that, depending on the values of the constitutive and geometric parameters of the structure, the resonance fields can be linearly or circularly polarized. Degenerate eigen oscillations are discovered. The fields of these oscillations exhibit different angular dependences in the near and far zones. On the basis of rigorous methods, the frequency characteristics of the structure, as well as the near- and far-field patterns at resonance frequencies, are calculated. An approximate analytical description of quasi-static resonances is proposed.

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1. FORMULATION OF THE PROBLEM AND THE METHOD OF SOLUTION

The axially symmetric problem of excitation of a metamaterial sphere by magnetic and electric dipoles oriented along the z axis and located on this axis at point $z = r_0$ beyond the sphere (Fig. 1) is investigated. The Gaussian system and the time dependence $\exp(i\omega t)$ of the fields are used.

In spherical coordinates (r, θ, φ), the spatial distributions of the permittivity and permeability are specified as follows:

$$\varepsilon(r) = \begin{cases} \varepsilon, & r < a, \\ 1, & r > a, \end{cases} \quad \mu(r) = \begin{cases} \mu, & r < a, \\ 1, & r > a, \end{cases} \tag{1}$$

where a is the radius of the sphere. Assume that the surface of the sphere $r = a$ exhibits anisotropic electric conductivity along spiral lines with constant lead angle ψ. The two-sided boundary conditions

$$E_\theta^+ = E_\theta^-, \quad E_\varphi^+ = E_\varphi^-, \quad E_\varphi \cos\psi = E_\theta \sin\psi, \tag{2}$$

$$\left(H_\varphi^+ - H_\varphi^-\right)\cos\psi = \left(H_\theta^+ - H_\theta^-\right)\sin\psi \tag{3}$$

are fulfilled on the surface $r = a$. Here, the plus and minus signs refer to the exterior, $r > a$, and interior, $r < a$, sides of the sphere, respectively. For the sake of definiteness, helical lines (loxodromes in the case considered) are assumed to be right-handed $(0 < \psi < \pi/2)$. Boundary conditions (2) and (3) describe wire helices in the case when the distance between the axes of neighboring conductors is much smaller than the wavelength and the gap dimension is within a certain interval [1].

The problem of diffraction of a plane wave by a chiral sphere filled with an ordinary magnetodielectric $(\varepsilon > 0, \mu > 0)$ is considered in studies [2, 3]. It was first

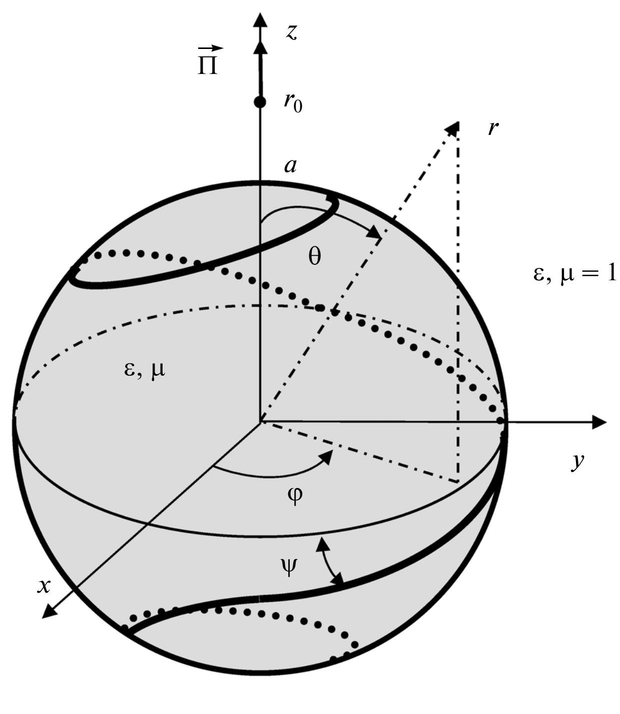

Fig. 1. Geometry of the problem.

shown in [2] that, when $\psi \ll 1$, resonance phenomena can be observed in a sphere small as compared to wavelength λ $(a \ll \lambda)$. The excitation of a sphere by a radial electric dipole is considered in [4]. The purpose of this study is to investigate the specific character of low-frequency resonances excited by radial magnetic and electric dipoles in a metamaterial sphere $(\varepsilon < 0, \mu < 0)$.

The primary field of dipoles can be expressed in terms of the only nonzero z components of the Hertz magnetic and electric vectors $\vec{\Pi}^{\rm m}$ and $\vec{\Pi}^{\rm e}$ [5]:

$$\Pi_z^{\rm m} = A_1 \frac{\exp(-ikR)}{R}, \quad \Pi_z^{\rm e} = A_2 \frac{\exp(-ikR)}{R}, \tag{4}$$

where

$$R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta}, \tag{5}$$

and $k = 2\pi/\lambda$ is the wavenumber in free space. Quantities A_1 and A_2 are proportional to the dipole moments of the sources. The knowledge of vectors $\vec{\Pi}^{\rm m}$ and $\vec{\Pi}^{\rm e}$ makes it possible to determine the electromagnetic field in free space [5]:

$$\begin{aligned} \vec{H} &= \operatorname{curl}\operatorname{curl}\vec{\Pi}^{\rm m} + ik\operatorname{curl}\vec{\Pi}^{\rm e}, \\ \vec{E} &= \operatorname{curl}\operatorname{curl}\vec{\Pi}^{\rm e} - ik\operatorname{curl}\vec{\Pi}^{\rm m}. \end{aligned} \tag{6}$$

We describe the axially symmetric electromagnetic fields $\left(\frac{\partial}{\partial\varphi} = 0\right)$ in the problem under consideration with the help of Hertz magnetic and electric potentials $U_1(r,\theta)$ and $U_2(r,\theta)$ [6]. To make the representation more compact, we use vector symbols in the notation of two-component quantities containing indices 1 and 2:

$$\vec{A} = \{A_1, A_2\}, \quad \vec{U} = \{U_1, U_2\}. \tag{7}$$

The Hertz potentials satisfy the equation

$$r^2\left[\frac{\partial^2\vec{U}}{\partial r^2} + k^2\varepsilon(r)\mu(r)\vec{U}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\vec{U}}{\partial\theta}\right) = 0. \tag{8}$$

The components of the electromagnetic field can be expressed through the Hertz potentials as

$$\begin{gathered} H_r = \left[\frac{\partial^2}{\partial r^2} + k^2\varepsilon(r)\mu(r)\right]U_1, \quad E_r = \left[\frac{\partial^2}{\partial r^2} + k^2\varepsilon(r)\mu(r)\right]U_2, \\ H_\theta = \frac{1}{r}\frac{\partial^2 U_1}{\partial r\partial\theta}, \quad E_\theta = \frac{1}{r}\frac{\partial^2 U_2}{\partial r\partial\theta}, \\ H_\varphi = \frac{-ik\varepsilon(r)}{r}\frac{\partial U_2}{\partial\theta}, \quad E_\varphi = \frac{ik\mu(r)}{r}\frac{\partial U_1}{\partial\theta}. \end{gathered} \tag{9}$$

Note that quantities U_1/r and U_2/r are known as the Debye potentials [7].

Primary field (4)–(6) is associated with Hertz potentials U_1^0 and U_2^0 determined from the formula [4]

$$\begin{aligned} \vec{U}^0 &= \frac{\vec{A}}{ikr_0^2}\sum_{m=1}^{\infty}(2m+1) \\ &\times P_m(\cos\theta)\begin{cases} j_m(kr)h_m^{(2)}(kr_0), & r < r_0, \\ j_m(kr_0)h_m^{(2)}(kr), & r > r_0, \end{cases} \end{aligned} \tag{10}$$

where $P_m(\cos\theta)$ are the Legendre polynomials, $j_m(kr)$ are the Riccati–Bessel functions, and $h_m^{(2)}(kr)$ are the Riccati–Hankel functions [8].

With the use of the Hertz potentials, an analytical solution to the formulated problem can be obtained by means of the standard method of separation of variables [4]. We present the final expressions for the wave fields. Let us introduce 2D vectors $\vec{L}^{(m)}$, $\vec{M}^{(m)}$, and $\vec{N}^{(m)}$ and scalar $W^{(m)}$:

$$\vec{L}^{(m)} = \left\{h_m^{(2)\prime}(ka)\sin\psi, \ ih_m^{(2)}(ka)\cos\psi\right\}, \tag{11}$$

$$\vec{M}^{(m)} = \left\{h_m^{(2)\prime}(ka)\sin\psi, \ -ih_m^{(2)}(ka)\cos\psi\right\}, \tag{12}$$

$$\vec{N}^{(m)} = \left\{\frac{n}{\mu}j_m'(kna)\sin\psi, \ ij_m(kna)\cos\psi\right\}, \tag{13}$$

$$\begin{aligned} W^{(m)} &= \varepsilon h_m^{(2)}(ka)j_m(kna) \\ &\times\left[\frac{n}{\varepsilon}h_m^{(2)}(ka)j_m'(kna) - h_m^{(2)\prime}(ka)j_m(kna)\right] \\ &\times\cos^2\psi + nh_m^{(2)\prime}(ka)j_m'(kna) \\ &\times\left[\frac{n}{\mu}h_m^{(2)}(ka)j_m'(kna) - h_m^{(2)\prime}(ka)j_m(kna)\right]\sin^2\psi, \end{aligned} \tag{14}$$

where

$$n = \sqrt{\varepsilon\mu}. \tag{15}$$

The prime in formulas (11)–(14) denotes differentiation of functions with respect to the argument.

The field inside the sphere is associated with the Hertz potentials

$$\vec{U} = \frac{\vec{A}}{ikr_0^2}\sum_{m=1}^{\infty}(2m+1)h_m^{(2)}(kr_0)\vec{B}^{(m)}j_m(nkr)P_m(\cos\theta), \quad r < a, \tag{16}$$

where

$$\vec{B}^{(m)} = \frac{i\left(\vec{A}, \vec{M}^{(m)}\right)}{W^{(m)}}\vec{N}^{(m)}. \tag{17}$$

In (17), $\left(\vec{A}, \vec{M}^{(m)}\right)$ denotes the scalar product

$$\left(\vec{A}, \vec{M}^{(m)}\right) = A_1M_1^{(m)} + A_2M_2^{(m)}. \tag{18}$$

The field in the exterior of the sphere ($r > a$) consists of two terms: incident and scattered fields $\vec{U}^0$ and $\vec{U}^s$. The scattered field is associated with the Hertz potential

$$\vec{U}^s = \frac{1}{ikr_0^2}\sum_{m=1}^{\infty}(2m+1)h_m^{(2)}(kr_0) \times \left(\vec{C}^{(m)} + \vec{D}^{(m)}\right)h_m^{(2)}(kr)P_m(\cos\theta), \quad r > a, \tag{19}$$

where

$$\vec{C}^{(m)} = \left\{-A_1\frac{j_m(ka)}{h_m^{(2)}(ka)}, \ -A_2\frac{j_m'(ka)}{h_m^{(2)\prime}(ka)}\right\}, \tag{20}$$

$$\vec{D}^{(m)} = \frac{i\left(\vec{A}, \vec{M}^{(m)}\right)nj_m'(nka)j_m(nra)}{h_m^{(2)\prime}(ka)h_m^{(2)}(ka)W^{(m)}}\vec{L}^{(m)}. \tag{21}$$

Note that the term from (19) that contains vector $\vec{C}^{(m)}$ is the field formed in the case of scattering by an isotropically conducting sphere. Formulas (11)–(21) formally are similar to the expressions obtained in [9], where the problem of excitation of an anisotropically conducting cylinder by electric- and magnetic-current filaments is considered. The corresponding formulas coincide accurate to the replacement of functions $j_m(kr), h_m^{(2)}(kr)$, and $P_m(\cos\theta)$ by Bessel functions $J_m(kr)$, Hankel functions $H_m^{(2)}(kr)$ and the trigonometric functions $\cos(m\varphi)$.

From relationships (9) and (19), we can obtain the following formulas for the azimuthal components of the electromagnetic field scattered by the sphere:

$$\begin{aligned} E_\varphi^s &= \frac{1}{r_0^2 r}\sum_{m=1}^{\infty}(2m+1)h_m^{(2)}(kr_0) \times \left(C_1^{(m)} + D_1^{(m)}\right)h_m^{(2)}(kr)P_m^1(\cos\theta), \quad r > a, \\ H_\varphi^s &= \frac{-1}{r_0^2 r}\sum_{m=1}^{\infty}(2m+1)h_m^{(2)}(kr_0) \times \left(C_2^{(m)} + D_2^{(m)}\right)h_m^{(2)}(kr)P_m^1(\cos\theta), \end{aligned} \tag{22}$$

where $P_m^1(\cos\theta)$ is the associated Legendre polynomial [8],

$$P_m^1(\cos\theta) = \frac{d}{d\theta}P_m(\cos\theta). \tag{23}$$

The far electric field scattered by the sphere has the form

$$E_\theta^s \sim F_\theta(\theta)\frac{\exp(-ikr)}{r}, \quad E_\varphi^s \sim F_\varphi(\theta)\frac{\exp(-ikr)}{r}. \tag{24}$$

The components of the vector scattering pattern can be determined from the formulas

$$\begin{aligned} F_\theta(\theta) &= \frac{1}{r_0^2}\sum_{m=1}^{\infty}i^{m-1}(2m+1)h_m^{(2)}(kr_0)\left(C_2^{(m)} + D_2^{(m)}\right)P_m^1(\cos\theta), \\ F_\varphi(\theta) &= \frac{-1}{r_0^2}\sum_{m=1}^{\infty}i^{m-1}(2m+1)h_m^{(2)}(kr_0)\left(C_1^{(m)} + D_1^{(m)}\right)P_m^1(\cos\theta). \end{aligned} \tag{25}$$

2. QUASI-STATIC RESONANCES

Let us consider only electrically small objects whose parameters satisfy the conditions

$$ka \ll 1, \quad nka \ll 1. \tag{26}$$

Expressions (16) and (19) for the Hertz potentials contain resonance denominators $W^{(m)}(ka)$ determined by formula (14). Let us analyze the frequency dependence of these denominators. Expression (14) is a complex function of the parameter ka and does not vanish for real values of ka. When conditions (26) are fulfilled, the real part of expression (14) substantially exceeds its imaginary part. The real parts of the denominators vanish at the points that are resonance frequencies. Thus, the equation for the resonance frequencies has the form

$$\mathrm{Re}\,W^{(m)}(ka) = 0. \tag{27}$$

At the resonance frequency, the only meridional harmonic $P_m(\cos\theta)$ dominates in decompositions (16) and (19).

To simplify Eq. (27), we use the known asymptotic expansions of the Riccati–Bessel and Riccati–Hankel functions for small values of the argument [8]

$$j_m(x) = \frac{x^{m+1}}{(2m+1)!!}\left[1 - \frac{x^2}{2(2m+3)} + \ldots\right], \tag{28}$$

$$h_m^{(2)}(x) = \frac{i(2m-1)!!}{x^m}\left[1 + \frac{x^2}{2(2m-1)} + \ldots\right]. \tag{29}$$

Assume that the condition

$$\left|m + \frac{m+1}{\mu}\right| \ll 1 \tag{30}$$

is fulfilled. This condition ensures the existence of low-frequency magnetic oscillations in a metamaterial sphere in the absence of a wire helix [10]. With the help of formulas (28)–(30), we obtain from Eq. (27) the following expression for resonance frequencies:

$$(ka)^2 = \frac{\left(m + \dfrac{m+1}{\mu}\right)\sin^2\psi}{\left(\dfrac{1}{2m-1} + \dfrac{\varepsilon}{2m+3}\right)\sin^2\psi + \left(\dfrac{1}{m} + \dfrac{\varepsilon}{m+1}\right)\cos^2\psi}. \tag{31}$$

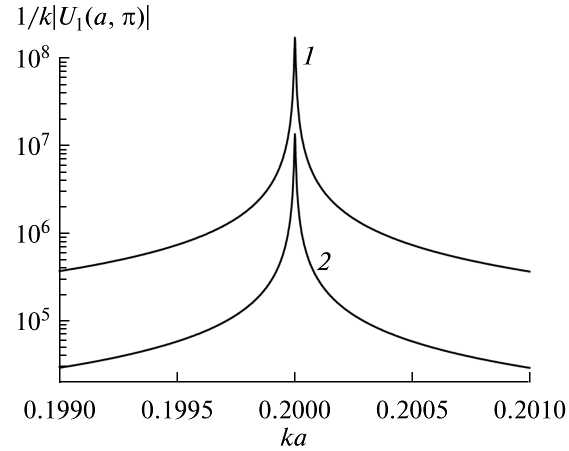

Fig. 2. Amplitude–frequency characteristic of a chiral sphere filled by a metamaterial. The results are obtained for $\varepsilon = -1.3$; $\mu = -1.3345454$ $ka = 0.2$; and $r_0 = 1.2a$. Curves *1* and *2* correspond to $A_1 = 1$, $A_2 = 0$ and $A_1 = 0$, $A_2 = 1$, respectively.

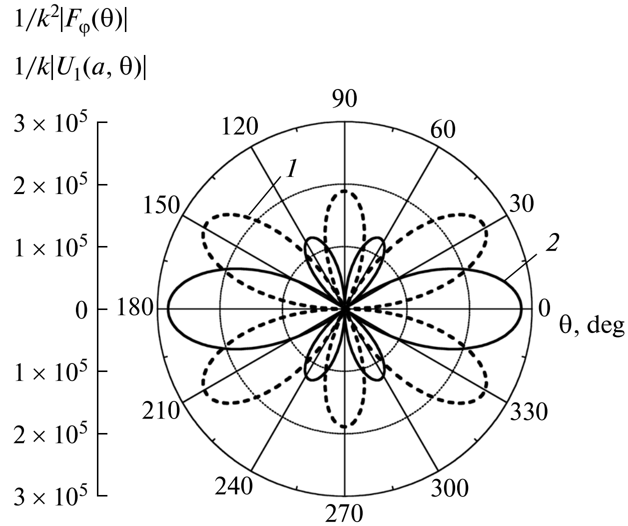

Fig. 3. Absolute values of the scattering pattern $|F_\varphi(\theta)|$ and the Hertz potential $|U_1(a, \theta)|$ at the resonance frequency $ka = 0.2$. The results are obtained for a chiral sphere filled by a metamaterial, $\varepsilon = -1.3$; $\mu = -1.3345454$; $r_0 = 1.2a$; $A_1 = 1$, and $A_2 = 0$. Curves *1* and *2* correspond to $|F_\varphi(\theta)|$ and $|U_1(a, \theta)|$, respectively. The scale for curve *2* is reduced by a factor of 50.

Recall that this expression is valid only under the condition $ka \ll 1$. Formula (31) can also be applied to a sphere made from an ordinary material $(\varepsilon > 0,\ \mu > 0)$. However, the expected result $ka \ll 1$ can be obtained in this case if the inequality

$$\psi \ll 1 \tag{32}$$

is fulfilled. The first term from the denominator in formula (31) can be neglected. Then, expression (31) becomes the formula for resonance frequencies obtained in [4].

In this study, we do not require that inequality (32) should be fulfilled as a necessary condition for the existence of low-frequency resonances. In the case of metamaterials, the right-hand side of relationship (31) can be small owing to condition (30). In this case, formula (31) can be applied to structures with arbitrary lead angles of conductivity lines. In particular, when $\psi = \pi/2$, expression (31) coincides with the formula for the resonance frequencies of quasi-static magnetic modes in a sphere with no lattice of conductors on its surface [11]. The coincidence is explained by the fact that the electric field of magnetic modes has only component E_φ. These modes do not interact with a wire lattice, because, at $\psi = \pi/2$, the conductors of the lattice are orthogonal to the electric field of a mode.

Let us analyze the amplitude–frequency characteristic (AFC) of the sphere. Here, the AFC is considered to mean the dependence of absolute values $|U_1(r,\theta)|$ and $|U_2(r,\theta)|$ of the potentials at the point $r = 0.99a$, $\theta = \pi$ on the dimensionless parameter ka, which is proportional to the frequency.

Figure 2 shows the AFCs for the case $\varepsilon = -1.3$, $\mu = -1.334, \ldots, \psi = 0.7$. The curves have a resonance character, and the resonance Q factor is a value on the order 10^4. Curves *1* and *2* correspond to two excitation conditions $\vec{A} = \{1,0\}$ and $\vec{A} = \{0,1\}$. It follows from the comparison of the curves that the efficiency of the excitation by an electric dipole is lower than the efficiency of the excitation by a magnetic dipole by an order of magnitude. The figure shows only the plots for the U_1 component. The values of the U_2 component are less than U_1 by an order of magnitude.

Absolute value $|F_\varphi|$ of the scattering pattern and the distribution of absolute value $|U_1|$ of the Hertz potential over the spherical surface at the resonance frequency are depicted in Fig. 3. These dependences can be described with a graphical accuracy by the functions $P_3^1(\cos\theta)$ and $P_3(\cos\theta)$, which correspond to the meridional harmonic with $m = 3$. It is seen from the figure that the directions of the maximal radiation correspond to the zeros of the Hertz potential (see (23)).

Figure 4 shows the distribution of absolute value $|U_1|$ of the Hertz potential along the radial coordinate in the direction $\theta = \pi$. The curve contains two sections $kr < 2$ and $kr > 10$, where the Hertz potential can be described by the functions $|U_1| \sim (kr)^{-3}$ and $|U_1| = \text{const}$. The first and second sections correspond to the near and far fields, respectively.

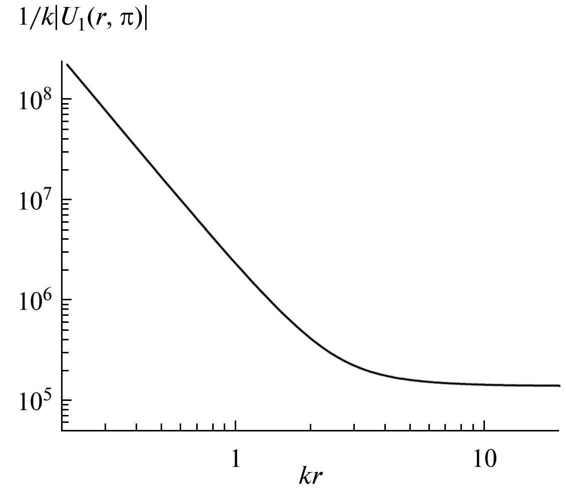

Fig. 4. Distribution of absolute value $|U_1(r,\pi)|$ of the Hertz potential along the radial coordinate. The results are obtained for a chiral sphere filled by a metamaterial, $\varepsilon = -1.3$; $\mu = -1.3345454$; $\psi = 0.7$; $r_0 = 1.2a$; $A_1 = 1$, and $A_2 = 0$ at the resonance frequency $ka = 0.2$.

3. THE FIELD IN THE QUASI-STATIC REGION

Let us study the spatial and polarization structures of the near resonance field. The discussed resonances are quasi-static ones: $ka \ll 1$. The electromagnetic field is localized in the region $kr \ll 1$ and rapidly decreases as the distance from the spherical surface grows. From formulas (11), (13), (16), (19), (28), and (29), we can obtain the following expressions for the Hertz potentials of eigen oscillations in the region $kr \ll 1$:

$$U_1 = \begin{cases} -\dfrac{i}{\mu}\left(\dfrac{r}{a}\right)^{m+1} P_m(\cos\theta)\sin\psi, & r < a, \\ -i\left(\dfrac{r}{a}\right)^{-m} P_m(\cos\theta)\sin\psi, & r > a, \end{cases} \tag{33}$$

$$U_2 = \begin{cases} \dfrac{ka}{m+1}\left(\dfrac{r}{a}\right)^{m+1} P_m(\cos\theta)\cos\psi, & r < a, \\ -\dfrac{ka}{m}\left(\dfrac{r}{a}\right)^{-m} P_m(\cos\theta)\cos\psi, & r > a. \end{cases} \tag{34}$$

If the quantities $\sin\psi$ and $\cos\psi$ are assumed to be on equal orders, the comparison of formulas (33) and (34) implies the inequality $|U_2| \ll |U_1|$. Therefore, the field structure of the eigen oscillation in the highest (in the parameter ka) order is the same as the field structure of magnetic modes.

Potentials (33) and (34) are associated with the electromagnetic fields

$$H_r = \begin{cases} -i\dfrac{m(m+1)}{\mu k^2}\left(\dfrac{r}{a}\right)^{m-1} P_m(\cos\theta)\sin\psi, & r < a, \\ -i\dfrac{m(m+1)}{k^2}\left(\dfrac{r}{a}\right)^{-m-2} P_m(\cos\theta)\sin\psi, & r > a, \end{cases} \tag{35}$$

$$H_\theta = \begin{cases} -\dfrac{i(m+1)}{\mu k^2}\left(\dfrac{r}{a}\right)^{m-1} P_m^1(\cos\theta)\sin\psi, & r < a, \\ \dfrac{im}{k^2}\left(\dfrac{r}{a}\right)^{-m-2} P_m^1(\cos\theta)\sin\psi, & r > a, \end{cases} \tag{36}$$

$$H_\varphi = \begin{cases} -\dfrac{i\varepsilon a^2}{m+1}\left(\dfrac{r}{a}\right)^{m-1} P_m^1(\cos\theta)\cos\psi, & r < a, \\ \dfrac{ia^2}{m}\left(\dfrac{r}{a}\right)^{-m-1} P_m^1(\cos\theta)\cos\psi, & r > a, \end{cases} \tag{37}$$

$$E_r = \begin{cases} \dfrac{ma}{k}\left(\dfrac{r}{a}\right)^{m-1} P_m(\cos\theta)\cos\psi, & r < a, \\ \dfrac{-(m+1)a}{k}\left(\dfrac{r}{a}\right)^{-m-2} P_m(\cos\theta)\cos\psi, & r > a, \end{cases} \tag{38}$$

$$E_\theta = \begin{cases} \dfrac{a}{k}\left(\dfrac{r}{a}\right)^{m-1} P_m^1(\cos\theta)\cos\psi, & r < a, \\ \dfrac{a}{k}\left(\dfrac{r}{a}\right)^{-m-2} P_m^1(\cos\theta)\cos\psi, & r > a, \end{cases} \tag{39}$$

$$E_\varphi = \begin{cases} \dfrac{a}{k}\left(\dfrac{r}{a}\right)^{m} P_m^1(\cos\theta)\sin\psi, & r < a, \\ \dfrac{a}{k}\left(\dfrac{r}{a}\right)^{-m-1} P_m^1(\cos\theta)\sin\psi, & r > a. \end{cases} \tag{40}$$

It follows from these formulas that the magnetic field prevails near the surface of the sphere: $|\vec{H}| \gg |\vec{E}|$. Components E_r, E_θ, and E_φ are commensurable, and component H_φ is substantially smaller than quantities H_r and H_θ, the difference being on the order $(ka)^2$.

We can conclude from expressions (39) and (40) that the electric field vector tangent to the spherical surfaces $r = \text{const}$ is linearly polarized and that

$$\frac{E_\theta}{E_\varphi} = \frac{a}{r}\cot\psi. \tag{41}$$

Thus, the electric-field lines on the spherical surface $r = \text{const}$ are loxodromes. It follows from formula (41) that, on the boundary of the sphere $r = a$, these lines are orthogonal to the conductors of the helix. As r grows, the lead angle of field lines decreases.

Since

$$\frac{E_r}{E_\theta} = \frac{H_r}{H_\theta} = \begin{cases} m\dfrac{P_m(\cos\theta)}{P_m^1(\cos\theta)}, & r < a, \\ -(m+1)\dfrac{P_m(\cos\theta)}{P_m^1(\cos\theta)}, & r > a, \end{cases} \tag{42}$$

the electric and magnetic field components lying in the azimuthal planes $\varphi = \mathrm{const}$ are linearly polarized and parallel. Field lines $r(\theta)$ of these 2D vector fields can be found from the differential equation

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{E_r}{E_\theta}. \quad (43)$$

Taking into account the relationship [8]

$$m(m+1)\sin\theta P_m(\cos\theta) = -\frac{d}{d\theta}\left[\sin\theta P_m^1(\cos\theta)\right], \quad (44)$$

we obtain from (42) and (43)

$$r(\theta) = \begin{cases} \alpha\left|\sin\theta P_m^1(\cos\theta)\right|^{\frac{-1}{(m+1)}}, & r < a, \\ \beta\left|\sin\theta P_m^1(\cos\theta)\right|^{\frac{1}{m}}, & r > a, \end{cases} \quad (45)$$

where α and β are arbitrary constants.

In particular, we obtain for $m = 1$

$$r(\theta) = \begin{cases} \dfrac{\alpha}{\sin\theta}, & r < a, \\ \beta\sin^2\theta, & r > a. \end{cases} \quad (46)$$

Beyond the sphere, the field lines are the same as those for an elementary dipole in the near zone. Inside the sphere, these lines form a set of straight lines parallel to the z axis [4]. The pattern of magnetic-field lines generally consists of $2m$ sectors whose boundaries are determined by the zeros of the function $P_m^1(\cos\theta)$. Inside each sector, the currents flowing along the conductors of the lattice have identical directions and the field lines form a system of embedded closed contours enclosing the point lying on the boundary of the sphere $r = a$ and having the angular coordinate that is the zero of the function $P_m(\cos\theta)$.

The eigenfrequencies of quasi-static oscillations can be found from the relationship that expresses the equality of the energies accumulated by the electric and magnetic fields of the resonance structure:

$$\int_0^\infty\int_0^\pi \mu(r)\left[\left|H_r(r,\theta)\right|^2 + \left|H_\theta(r,\theta)\right|^2\right] r^2 \sin\theta\, dr\, d\theta$$
$$= \int_0^\infty\int_0^\pi \varepsilon(r)\left[\left|E_r(r,\theta)\right|^2 + \left|E_\theta(r,\theta)\right|^2 + \left|E_\varphi(r,\theta)\right|^2\right] \quad (47)$$
$$\times r^2 \sin\theta\, dr\, d\theta.$$

Taking into account formulas (35), (36) and (38)–(40), we obtain from (47) Eq. (31). Note that, in the physical sense, this equation is an analog of the Thomson formula $\omega^2 = 1/LC$, which determines the resonance frequency of an LC contour.

4. CIRCULARLY POLARIZED OSCILLATIONS

The interest in an anisotropically conducting sphere has been aroused mainly owing to its property to selectively respond to circularly polarized fields with different directions of rotation [2–4]. The application of metamaterials opens new possibilities of realizing devices that use quasi-static resonances with a circularly polarized field. Let us find the conditions under which the scattered field at the resonance frequency is circularly polarized.

Let $F_\theta^{(m)}(\theta)$ and $F_\varphi^{(m)}(\theta)$ denote the contributions of harmonics $P_m^1(\cos\theta)$ to vector scattering patterns $F_\theta(\theta)$ and $F_\varphi(\theta)$ determined by formulas (25). We obtain from formulas (11), (21), and (25)

$$\frac{F_\varphi^{(m)}}{F_\theta^{(m)}} = -\frac{D_1^{(m)}}{D_2^{(m)}} = -\frac{L_1^{(m)}}{L_2^{(m)}}$$
$$= i\frac{h_m^{(2)\prime}(ka)}{h_m^{(2)}(ka)}\tan\psi \approx -i\frac{m\tan\psi}{ka}. \quad (48)$$

It follows from expression (48) that, at the frequency satisfying the condition

$$ka = m\tan\psi, \quad (49)$$

the scattered field harmonic $P_m^1(\cos\theta)$ is counterclockwise-polarized, $F_\varphi^{(m)}(\theta) = -iF_\theta^{(m)}(\theta)$, and this harmonic is not excited by clockwise-polarized field $\vec{A} = \{1, i\}$, because $\left(\vec{M}, \vec{A}\right) = 0$.

In the case under consideration $(ka \ll 1)$, condition (49) can be fulfilled only when $\psi \ll 1$. If small quantities ka and ψ are coupled by relationship (49), formulas (35)–(37) imply that components H_r, H_θ, and H_φ are commensurable. Some of the formulas were derived in the previous section with the case $\psi \ll 1$ disregarded. As a result, formula (31), which is derived with disregard of component H_φ (see (47)) is inaccurate.

Consider in more detail the case when the lead angle is small and the constitutive parameters satisfy the condition

$$\varepsilon = \mu \approx -1 - \frac{1}{m}. \quad (50)$$

It follows from formula (14) and (27) that the equation for the determination of resonance frequencies at $\varepsilon = \mu$ has the form

$$\mathrm{Re}\left\{\begin{array}{c} \left[\mu h_m^{(2)}(ka) j_m(nka)\cos^2\psi\right. \\ \left. +\ n h_m^{(2)\prime}(ka) j_m'(nka)\sin^2\psi\right] \\ \times\left[\dfrac{n}{\mu} h_m^{(2)}(ka) j_m'(nka) - h_m^{(2)\prime}(ka) j_m(nka)\right] \end{array}\right\} = 0. \quad (51)$$

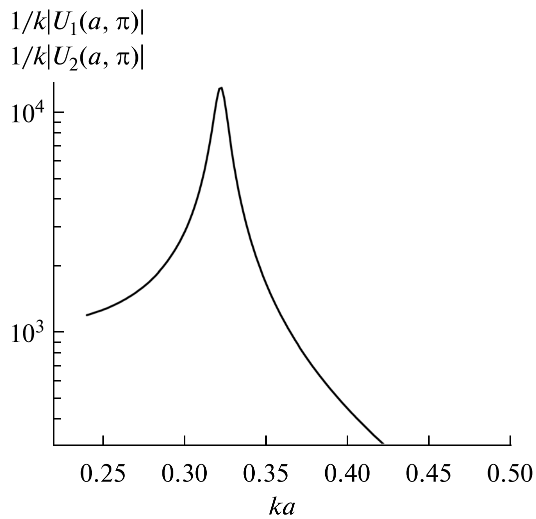

Fig. 5. Amplitude–frequency characteristic of a chiral sphere filled by a metamaterial. The results are obtained for $\varepsilon = \mu = -1.51$; $\psi = 0.1647$; $r_0 = 1.2a$; $A_1 = 1$, and $A_2 = -i$ at the resonance frequency $ka = 0.3217$.

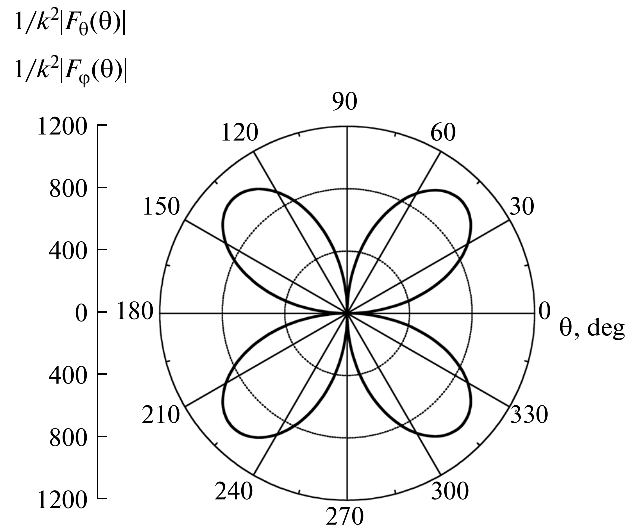

Fig. 6. Absolute values of the scattering patterns of a chiral sphere filled by a metamaterial. The results are obtained for $\varepsilon = \mu = -1.51$; $\psi = 0.1647$; $r_0 = 1.2a$; $A_1 = 1$, and $A_2 = -i$ at the resonance frequency $ka = 0.3217$.

With allowance for asymptotic expansions (28) and (29), vanishing of the first factor from (51) yields the equation

$$\mu(ka)^2\cos^2\psi - m(m+1)\sin^2\psi = 0. \quad (52)$$

This equation has a real solution only in the case of ordinary media ($\varepsilon > 0$, $\mu > 0$):

$$ka = \sqrt{\frac{m(m+1)}{\mu}}\tan\psi. \quad (53)$$

Expression (53) coincides with formula (24) from [4] under the assumption that $\varepsilon = \mu$. For metamaterials ($\varepsilon < 0$, $\mu < 0$), Eq. (52) has no real solutions.

The second factor from (51) yields the equation

$$\operatorname{Im}\left[\frac{n}{\mu}h_m^{(2)}(ka)j_m'(nka) - h_m^{(2)\prime}(ka)j_m(nka)\right] = 0. \quad (54)$$

Expression (54) coincides with the dispersion equation for the resonance frequencies of both magnetic and electric modes in a sphere with no lattice of conductors on its surface [11]. Thus, when $\varepsilon = \mu$, we observe degeneration of the axially symmetric modes each of which contains only three components of the electromagnetic field: E_φ, H_r, H_θ and H_φ, E_r, E_θ. The presence of a lattice of conductors on the sphere's surface results in the formation of a linear combination of these modes at a resonance frequency. At $r = a$, this combination satisfies the boundary conditions $E_\varphi\cos\psi = E_\theta\sin\psi$. As follows from expressions (36), (37), and (50), the jumps of the tangent components of the magnetic field on the sphere's surface $H_\theta^+ - H_\theta^-$ and $H_\varphi^+ - H_\varphi^-$ are negligibly small. This means the absence of currents flowing along the conductors of the lattice under the resonance conditions.

Taking into account asymptotic expansions (28) and (29) and condition (50), we can obtain from Eq. (54) the following expression for resonance frequencies:

$$(ka)^2 = -\frac{m^3(2m-1)(2m+3)}{(m+1)(2m+1)}\left(\mu + \frac{m+1}{m}\right). \quad (55)$$

Thus, the resonance frequency does not depend on angle ψ. However, the relationship between components $F_\varphi^{(m)}$ and $F_\theta^{(m)}$ depends on ψ. Choosing the value of ψ according to formula (49), we can provide for the circular polarization of the scattered field. For example, for the harmonic $P_2^1(\cos\theta)$, the circular polarization is realized for the parameters $\varepsilon = \mu = -1.51$ and $\psi = 0.16225$ at the resonance frequency $ka = 0.3217$.

Figures 5–7 illustrate rigorous computation of the AFC and scattering pattern for a sphere with the parameters $\varepsilon = \mu = -1.51$. Lead angle ψ is selected so as to provide for a strictly circularly polarized scattered field at the resonance frequency. Quantities ψ and ka turn out to be rather close to the values obtained from approximate formulas (49) and (55). The patterns from Figs. 6 and 7 correspond to the clockwise and counterclockwise excitations $A = \{1, -i\}$ and $A = \{1, i\}$, respectively. It is seen from Fig. 7 that, in the latter case, there is no resonance. In this situation, scattered field (19) is mainly determined by the nonresonance term containing vector $\vec{C}^{(1)}$ and differs from the pri-

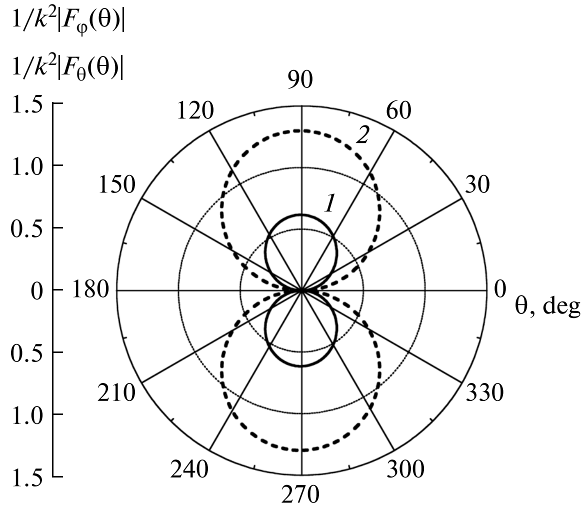

Fig. 7. Absolute values $|F_\varphi(\theta)|$ and $|F_\theta(\theta)|$ of the scattering patterns of a chiral sphere filled by a metamaterial. The results are obtained for $\varepsilon = \mu = -1.51$; $\psi = 0.1647$; $r_0 = 1.2a$; $A_1 = 1$, and $A_2 = -l$ at the resonance frequency $ka = 0.3217$; Curves *1* and *2* correspond to $|F_\varphi(\theta)|$ and $|F_\theta(\theta)|$.

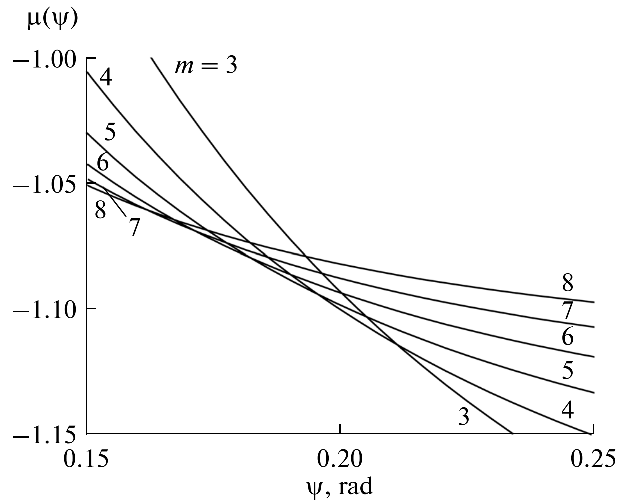

Fig. 8. Dispersion curves $\mu(\psi)$ for a chiral sphere filled by a metamaterial. The results are obtained for $\varepsilon = -4.0$; $ka = 0.2$, and various values of index m.

mary field of elementary dipoles only slightly. It follows from Fig. 7 that $|F_\theta(\theta)| \approx 2|F_\varphi(\theta)|$. This result matches the formula

$$\frac{C_2^{(1)}}{C_1^{(1)}} = -2\frac{A_2}{A_1}, \tag{56}$$

following from expressions (20), (28), and (29).

Note that, in the case of ordinary materials ($\varepsilon > 0$ and $\mu > 0$), low-frequency resonances can be interpreted as the resonances of the wave of a current flowing along a helical conductor [2]. When $\psi \ll 1$, total length l of the loxodrome between the poles $\theta = 0$ and $\theta = \pi$ is $l = \pi a/\psi$, and we have $l \sim \lambda/2$ for the circularly polarized oscillation with the index $m = 1$. In the case considered in this section, the resonance frequency is determined by formula (55) rather than being related with the loxodrome length.

5. DEGENERATE OSCILLATIONS

The case may occur such that parameters ε, μ, ψ, and ka have simulataneously resonance values for two different values of index m. Figure 8 displays the set of dispersion curves describing the relation between the resonance values of ψ and μ at $ka = 0$ and $\varepsilon = -4$. Different curves are associated with different values of index m. The curves from Fig. 8 are plotted according to formula (31). It is seen that all of the curves intersect, a circumstance that corresponds to the degeneration effect. Thus, at the point $\psi = 0.197, \ldots, \mu = -1.086\ldots$, the curves corresponding to the indices $m = 3$ and $m = 7$ intersect. Figures 9 and 10 depict the distributions of potential U_1 over the surface of the sphere $(r = a)$ and in the far zone $(kr \gg 1)$ for the parameter values that lead to the degeneration of the harmonics $P_3(\cos\theta)$ and $P_7(\cos\theta)$. It follows from Figs. 9 and 10 that the harmonics $P_7(\cos\theta)$ and $P_3(\cos\theta)$ dominate on the spherical surface and in the far zone, respectively. The distribution of the absolute value of Hertz potential U_1 along the radius in the direction $\theta = \pi$ is depicted in Fig. 11. It is seen that, in the near zone, the curve contains two sec-

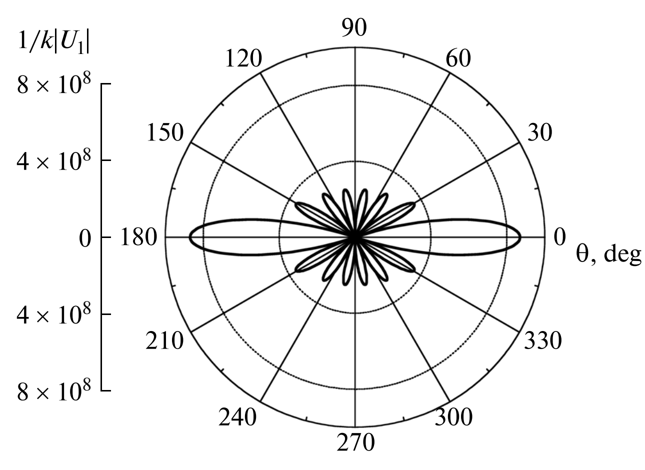

Fig. 9. Absolute value $|U_1(r, \theta)|$ of the Hertz potential on the surface of a chiral sphere $(r = a)$ filled by a metamaterial. The results are obtained for $\varepsilon = -4.0$; $ka = 0.2$; $\mu = -1.08618467$; $\psi = 0.19708157312$; $r_0 = 3a$; $A_1 = 1$, and $A_2 = 0$.

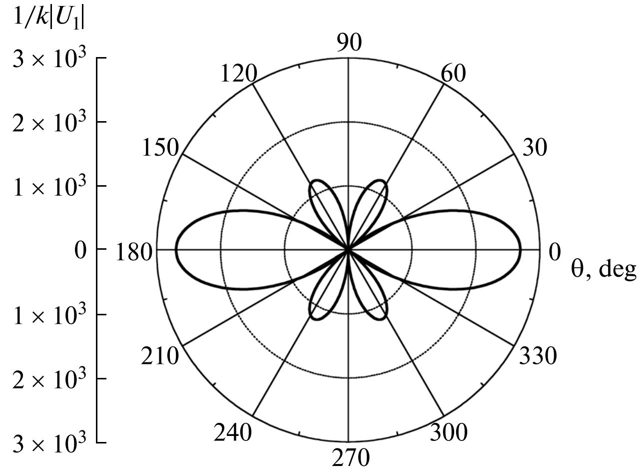

Fig. 10. Absolute value $|U_1(r,\theta)|$ of the Hertz potential in the far zone of a chiral sphere($r = 12a$) filled by a metamaterial. The results are obtained for $\varepsilon = -4.0$; $ka = 0.2$; $\mu = -1.08618467$; $\psi = 0.19708157312$; $r_0 = 3a$; $A_1 = 1$, and $A_2 = 0$.

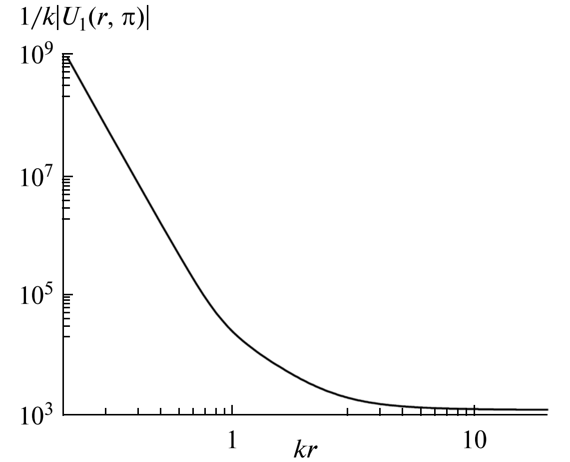

Fig. 11. The distribution of absolute value $|U_1(r,\pi)|$ of the Hertz potential along the radial coordinate for a chiral sphere filled by a metamaterial. The results are obtained for $\varepsilon = -4.0$; $\mu = -1.08618467$; $\psi = 0.19708157312$; $r_0 = 3.0a$; $A_1 = 1$, and $A_2 = 0$ at the resonance frequency $ka = 0.2$.

tions in which it decreases as $|U_1| \sim (kr)^{-7}$ and $|U_1| \sim (kr)^{-3}$, respectively.

In the considered example, lead angle ψ is small and $\psi \sim ka$. Therefore, potentials U_1 and U_2 are commensurable, and the plots from Figs. 9–11 qualitatively describe the behavior of potential U_2 as well.

Thus, quasi-static resonances have been revealed and investigated. Their characteristics and the region of existence substantially differ from the characteristics of resonances in an anisotropically conducting sphere filled with an ordinary magnetodielectric. It has been shown that, depending on the values of the constitutive and geometric parameters of the structure, the resonance fields can be linearly or circularly polarized. The values of parameters providing for the chiral properties of the electrodynamic structure have been found. These chiral properties are as follows: the scattered field is circularly polarized and a resonance occurs only for the specific direction of the rotation of the incident field polarization. The effect of eigen oscillation degeneration has been discovered. This effect consists in that different meridional harmonics of the resonance field dominate on the sphere's surface and in the far zone. An approximate analytic description of quasi-static resonances has been proposed.

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