Quasi-periodic Oscillations in the System of Three Chaotic Oscillators

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Abstract. The dynamics of three coupled chaotic Rössler systems is considered. We discuss scenarios for the evolution of different types of regimes. The possibility of two- and three-frequency quasi-periodicity is shown. We considered the occurrence of resonanses on three-frequency torus, which leads to two-freqiency quasi-periodic regimes. The illustrations in the form of charts of the Lyapunov exponents, phase portraits of attractors plotted in the Poincare section and bifurcation diagrams are presented. We discuss the type of quasi-periodic bifurcation in the system.

Keywords: chaotic oscillations, quasi-periodic oscillations, invariant tori, bifurcation.

1 Introduction

The problem related to oscillations of coupled oscillators of different nature remains the focus of researchers in different fields of physics, chemistry, biology. The examples are radio-electronic oscillators , Josephson contacts, ion traps [1-4], etc. One of the interesting aspects is the problem of synchronization of chaotic systems. The traditional approach in this case is to study the regimes for which the dynamics is chaotic, although it may be both synchronous and asynchronous [4,5]. In the works [4,5] the corresponding structure of the parameter plane (frequency detuning – parameter of coupling) is studied for two coupled Rossler oscillators. Also they pointed to the existence of different windows of periodic regimes. We consider here another situation when the dynamics of coupled chaotic systems becomes quasi-periodic. This is explained by a stabilizing effect of dissipative coupling, which, however, retains some basic oscillatory rhythm of the individual oscillators. We will discuss this problem by the example of three chaotic Rössler oscillators. In this case we found not only a two-frequency quasi-periodic regimes, but also three-frequency quasi-periodic regimes.

2 Three Chaotic Oscillators

Let us consider the system of three coupled Rössler oscillators:

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$$\begin{aligned}
\dot{x}_1 &= -y_1 - z_1, \\
\dot{y}_1 &= x_1 + py_1 + \mu(y_2 - y_1), \\
\dot{z}_1 &= q + (x_1 - r)z_1, \\
\dot{x}_2 &= -(1-\Delta_1)y_2 - z_2, \\
\dot{y}_2 &= (1-\Delta_1)x_2 + py_2 + \mu(y_1 + y_3 - 2y_2), \\
\dot{z}_2 &= q + (x_2 - r)z_2, \\
\dot{x}_3 &= -(1-\Delta_2)y_3 - z_3, \\
\dot{y}_3 &= (1-\Delta_2)x_3 + py_3 + \mu(y_2 - y_3), \\
\dot{z}_3 &= q + (x_3 - r)z_3.
\end{aligned} \tag{1}$$

Here Δ_1 is the frequency detuning between the first and second oscillators and Δ_2 is the frequency detuning between the first and third oscillators. We fix parameters p=0.15, q=0.4 and r=8.5. This corresponds to the chaotic regime in individual subsystems.

Let us disscus the question of how the regimes of different types are embedded in parameter space. For this, we use the method of the charts of Lyapumov exponents [6-10]. We calculate the spectrum of Lyapunov exponents at each grid point on the parameter plane. Then we color these points in accordance with its signature. The corresponding chart is given in Fig.1. It is plotted on the (Δ_1, μ) plane. The periodic regimes lettered by P, two- and three-frequency quasi-periodic regimes T2 and T3 (with one and two zero Lyapunov exponents respectively), regimes of chaos C (with one positive Lyapunov exponent), regimes of hyperchaos HC2 and HC3 (with two and three positive Lyapunov exponents respectively) are marked by different colors. Regime of “amplitude death” AD is responsible for disappearance of oscillations due to their suppression of a dissipative coupling. The color legend is at the right of the figure.

The phase portraits of attractors plotted in the Poincaré section are shown in Fig. 2 (The Poincaré section is defined by relations y=0 and x>0). Two-frequency torus T2 exists for large values of coupling. In this case Poincaré section is the invariant curve close to a circle. Three-frequency torus T3 arises softly from this invariant curve as the parameter of coupling is decreased. One can see a very intricately shaped invariant curve with a further decrease of coupling. This invariant curve corresponds to one of the possible two-frequency resonant tori T_R2. Note that the number of resonance windows is sufficiently large for these values of the frequency detuning. At small coupling the tori are destroyed with the appearance of chaos and hyperchaos.

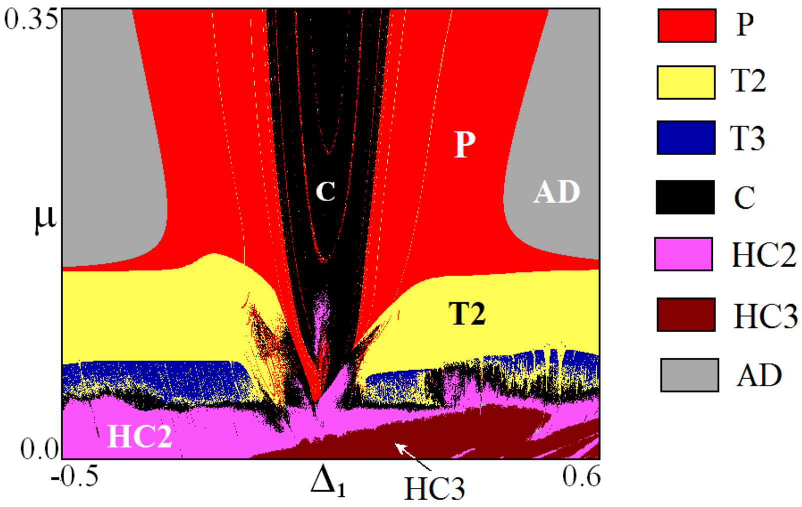

Fig. 1. Chart of Lyapunov exponent for the system (1), Δ_2=0.05

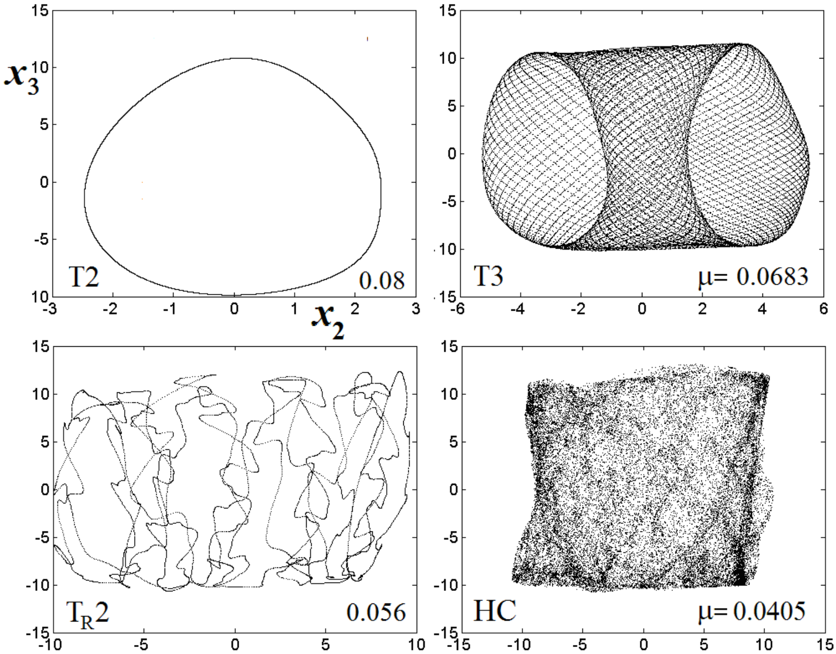

Fig. 2. Phase portraits of attractors at Poincaré section, Δ_1=0.19 and Δ_2=0.05

Fig. 3 shows the bifurcation diagram for the attractor in the chosen Poincare section versus the coupling parameter. This Figure illustrates the bifurcations responsible for the arising of invariant tori of different dimensions. Neumark-Sacker bifurcation of two-frequency torus occurs at the point NS. The windows of resonant limit cycles can be observed in the region of smaller values of coupling. The diagram widens sharply at the point QH. This is a point of quasi-periodic Hopf bifurcation [10-11], where three-frequency torus arises softly from two-frequency torus. Thus, the upper boundary of the region of three-frequency tori corresponds to the quasi-periodic Hopf bifurcation.

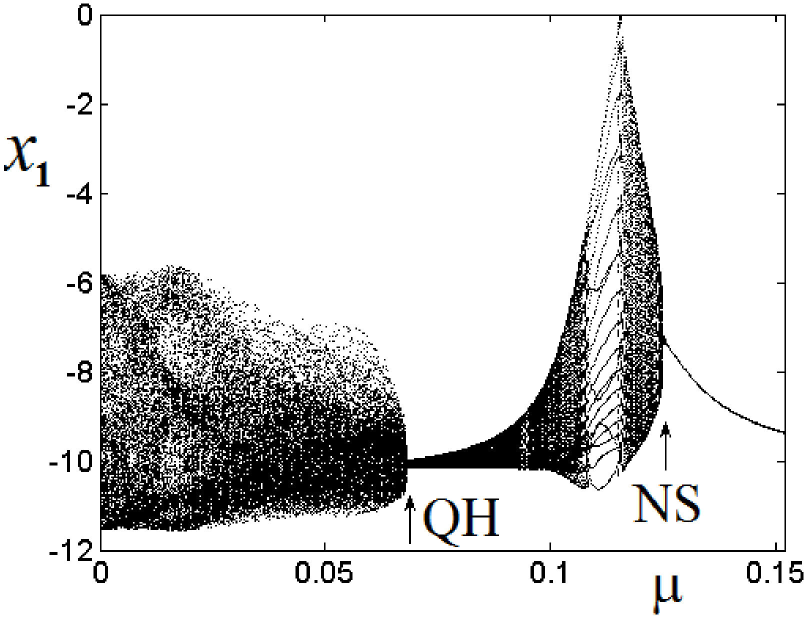

Fig. 3. Bifurcation diagram of system (1), Δ_1=0.19 and Δ_2=0.05

Quasi-periodic Hopf bifurcation QH is clearly visible in the enlarged fragment of the chart of Lyapunov exponents (Fig. 4). It is a boundary between three-frequency and two-frequency regions. One can see also a variety of tongues of two-frequency resonant tori. They have characteristic rounded tops, which are located along the QH line slightly above it. The tongues are destroyed with the appearance of chaos as the coupling decreases. Chaotic dynamics of individual oscillators is responsible for this. Note, the transition region from the three-frequency quasi-periodicity to chaos has a complex organization. In this region one can see a large number of resonances, which are much more numerous than in the case of regular oscillators [9].

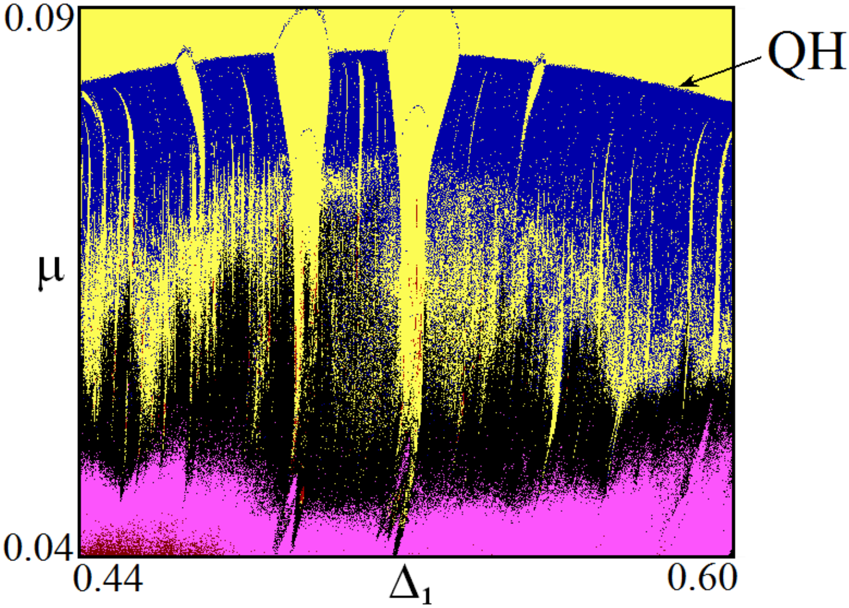

Fig. 4. Enlarged fragment of the parameter plane from Fig. 1, QH is a quasi-periodic Hopf bifurcation

3 Conclude

Thus, the effect of dissipative coupling on the chaotic oscillators can lead not only to the chaotic synchronization and appearance of periodic regimes, but also to the appearance of two- and three-frequency quasi-periodic oscillations. And quasi-periodic Hopf bifurcation is responsible for this. The reason is probably that chaotic regime is characterized by presence of large number of unstable limit cycles [4]. Adding of coupling can stabilize these cycles and this leads to appearance of the set of the resonant tori of different types in the dynamics of the system. With increasing of the number of chaotic oscillators the tori of higher and higher dimensions can be observed. We can expected, that this behavior would be typical for other chaotic systems.

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