Physics Letters A 378 (2014) 153–157

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Quasi-periodic bifurcations and "amplitude death" in low-dimensional ensemble of van der Pol oscillators

Yu.P. Emelianova [a,*], A.P. Kuznetsov [b], L.V. Turukina [b,c]

[a] Department of Electronics and Instrumentation, Saratov State Technical University, Polytechnicheskaya 77, Saratov 410054, Russian Federation
[b] Kotel'nikov's Institute of Radio-Engineering and Electronics of RAS, Saratov Branch, Zelyenaya 38, Saratov 410019, Russian Federation
[c] Institute for Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24-25, 14476 Potsdam, Germany

ARTICLE INFO

Article history:
Received 24 January 2013
Received in revised form 21 October 2013
Accepted 28 October 2013
Available online 31 October 2013
Communicated by A.P. Fordy

Keywords:
Chain of van der Pol oscillators
Bifurcations of tori
Amplitude death
Broadband synchronization

ABSTRACT

The dynamics of the four dissipatively coupled van der Pol oscillators is considered. Lyapunov chart is presented in the parameter plane. Its arrangement is discussed. We discuss the bifurcations of tori in the system at large frequency detuning of the oscillators. Here are quasi-periodic saddle-node, Hopf and Neimark–Sacker bifurcations. The effect of increase of the threshold for the "amplitude death" regime and the possibilities of complete and partial broadband synchronization are revealed.

1. Introduction

Synchronization of the ensembles of oscillators is a fundamental problem. It is problem of a general theoretical interest and it is important from the point of view of its application in biophysics, laser physics, electronics, chemistry, etc. [1–5]. The possible types of dynamical regimes of interacting oscillators are well studied for a small number of oscillators, or, in contrary, for the large number of oscillators. It should be noted that the dynamics of the chain of the oscillators has a number of specific features in comparison with the dynamics of the network or ring, consisted of the same number of elements. This is due to the fact that in the chain elements are not equal, in contrast to the network or ring. They can have a different number of neighbors depending on the position in the chain.

In present paper we consider the chain of four van der Pol oscillators. We discuss the main types of dynamical regimes typical for this model. At the same time, we do not use any approximations (i.e., the Landau–Stuart equations, the phase equations), except for some estimates. This means that both the limit cycles and the invariant tori of different dimensions are the attractors of this system. The presence of tori leads to the possibility of quasi-periodic bifurcations. We discuss the main types of these bifurcations in the system of four coupled van der Pol oscillators. Unfortunately, numerical algorithms for searching of the quasi-periodic bifurcations are very complicated and generally are under development [6,7]. However, the behavior of the Lyapunov exponents allows the identifications of the bifurcations.

The cases of a chain of two and three oscillators were discussed in detail in the works [8–10]. Several typical areas can be distinguished in the parameter plane (the frequency detuning and the coupling parameter) for the system of two coupled oscillators [1–3,8–10]. First, there is a synchronization region with frequency ratio of 1 : 1. Second, there is a region of quasi-periodic regimes. The set of synchronization tongues with other rational frequency ratios is embedded to this region. Another typical region is a region of the "amplitude death". It is located at sufficiently large frequency detuning values. In this case, the following condition must be satisfied: the coupling parameter is greater than the excitation parameter [1,11]. In this case the dissipative coupling compensates for excitation of the oscillators. Another regime is observed for the non-identical oscillators [9,12]. In this case the region of periodic regimes arises between the "amplitude death" region and the region of the quasi-periodic regimes. It is observed for arbitrarily large values of the frequency detuning. This region arises because the first oscillator dominates over the second oscillator, which is suppressed by a dissipative coupling. These regimes were called *broadband synchronization* in [12].

There are two types of quasi-periodic oscillations for the case of three dissipatively coupled oscillators. These regimes correspond to the two-frequency and three-frequency tori [13]. The resonances are observed as the parameter of the frequency detuning is small.

* Corresponding author. Tel.: +7 9172 021488.
E-mail addresses: yuliaem@gmail.com (Yu.P. Emelianova), apkuz@rambler.ru (A.P. Kuznetsov), lvtur@rambler.ru (L.V. Turukina).

0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.10.049

As a result, there is a set of tongues of two-frequency tori embedded in the region of three-frequency tori. For large frequency detuning the synchronization picture becomes simpler. The three-frequency tori arise at small coupling. And the two-frequency tori arise as the coupling parameter increases. The "amplitude death" region is observed with a further increase of the coupling parameter.

Increasing the number of oscillators to four leads to a number of new effects. They are the subject of the present paper. We discuss a cascade of bifurcations of periodic and quasi-periodic regimes. We show that the region of "amplitude death" is observed at essentially larger values of coupling parameter compared to the case of two and three coupled oscillators. Also, we show that the region of *complete broadband synchronization* arises in the case of four identical oscillators. This region separates the region of "amplitude death" and the region of two-frequency tori.

2. Structure of the parameter plane and quasi-periodic bifurcations

Let us consider a chain of four dissipatively coupled van der Pol oscillators.

$$\begin{aligned}
&\ddot{x} - (\lambda - x^2)\dot{x} + x + \mu_1(\dot{x} - \dot{y}) = 0,\\
&\ddot{y} - (\lambda - y^2)\dot{y} + (1 + \Delta_1)y + \mu_1(\dot{y} - \dot{x}) + \mu_2(\dot{y} - \dot{z}) = 0,\\
&\ddot{z} - (\lambda - z^2)\dot{z} + (1 + \Delta_2)z + \mu_2(\dot{z} - \dot{y}) + \mu_3(\dot{z} - \dot{w}) = 0,\\
&\ddot{w} - (\lambda - w^2)\dot{w} + (1 + \Delta_3)w + \mu_3(\dot{w} - \dot{z}) = 0.
\end{aligned} \tag{1}$$

Here λ is the control parameter responsible for excitation of the partial oscillators; Δ_1, Δ_2, and Δ_3 are frequency detuning between the second and the first oscillators, the third and the first oscillators, the forth and the first oscillators respectively; μ_i are the parameters of the dissipative coupling. Then we set all the coupling parameters equal to μ, i.e. $\mu_1 = \mu_2 = \mu_3 = \mu$. This allows us to more fully understand the mechanisms of suppression of the oscillations of different oscillators in the chain.

In order to get a general conception about the structure of the parameter plane, we use the method of *the charts of Lyapunov exponents* [13,14]. According to this method, we calculate the Lyapunov exponents L_i at each grid point on the parameter plane. Then we define the type of the regime in the system in accordance with the values of the Lyapunov exponents[1]:

1. P is the region of the limit cycle. The Lyapunov exponents are $L_1 = 0$, $L_2 < 0$, $L_3 < 0$, $L_4 < 0$, $L_5 < 0$;
2. T_2 is the region of the two-frequency torus. The Lyapunov exponents are $L_1 = L_2 = 0$, $L_3 < 0$, $L_4 < 0$, $L_5 < 0$;
3. T_3 is the region of the three-frequency torus. The Lyapunov exponents are $L_1 = L_2 = L_3 = 0$, $L_4 < 0$, $L_5 < 0$;
4. T_4 is the region of the four-frequency torus. The Lyapunov exponents are $L_1 = L_2 = L_3 = L_4 = 0$, $L_5 < 0$;
5. C is the region of the chaotic attractor. The Lyapunov exponents are $L_1 > 0$, $L_2 < 0$, $L_3 < 0$, $L_4 < 0$, $L_5 < 0$.

Then we color the points on the parameter plane in accordance with the type of the regime.

The chart of the Lyapunov exponents for the system (1) on the parameter plane (Δ_1, μ) is shown in Fig. 1a. It is plotted for $\lambda = 0.1$. Other parameters are $\Delta_2 = 0.03$, $\Delta_3 = 0.1$. That is, the frequencies of the first, the third and the fourth oscillators are sufficiently close to each other. When the coupling parameter is small and frequency detuning Δ_1 is near zero, we can observe the regions of resonant tori of different dimensions, Fig. 1a. In particular, there are two tongues of the three-frequency tori which are embedded in the region of the four-frequency tori. The tops of these tongues lie on the axis Δ_1 and satisfy two resonance conditions:

$$\Delta_1 = 0 \quad \text{or} \quad \Delta_1 = \Delta_2. \tag{2}$$

These relations correspond to the resonances when the frequency of the second oscillator coincides with the frequency of the first or the third oscillator. The values of the frequency detuning Δ_1 that correspond to the conditions (2) are marked in Fig. 1a with arrows. The regions of two-frequency tori also look like the synchronization tongues. But they all have threshold values for the coupling parameter. The tops of some of these tongues are destroyed with the appearance of the chaos.

Let us consider the chart of the Lyapunov exponents plotted for a wider parameter range, Fig. 1b. On this chart one can see the region of the "amplitude death". It is indicated as AD. As we note in the introduction, it is useful to investigate the bifurcations at large values of the frequency detuning Δ_1. In this case, one can observe the bifurcations for tori of higher and higher dimension as coupling parameter μ decreases.

Let us discuss the bifurcations in more detail. To do this, we turn to the graphs of the five largest Lyapunov exponents L_i of the system (1). It is presented in Fig. 2. The exponents are plotted versus the coupling parameter μ along the line $\Delta_1 = 2$.

One can see, that for large value of the coupling parameter μ all exponents in Fig. 2 are negative. This corresponds to the "amplitude death" regime. At the point H ($\mu \approx 0.27$) the first exponent L_1 becomes equal to zero. It is the point of the Hopf bifurcation. The stable (multidimensional) limit cycle arises at this point.

As the coupling parameter μ decreases the second exponent L_2 becomes equal to zero at point NS. This corresponds to the

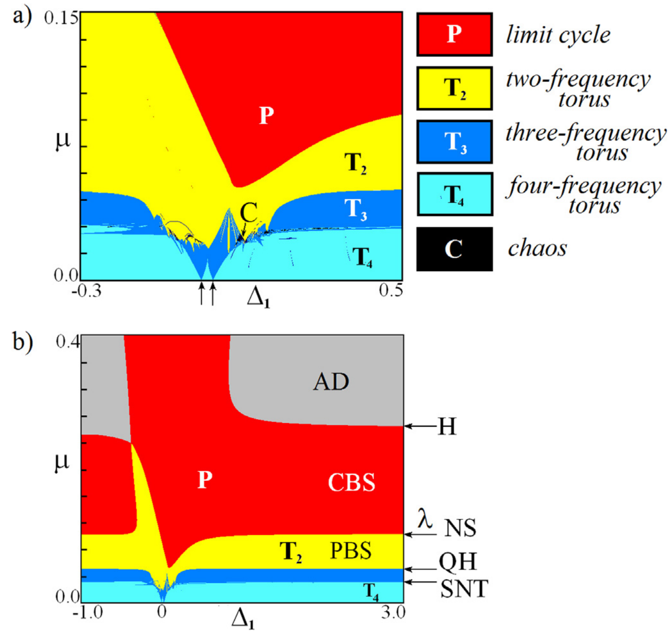

Fig. 1. (Color online.) Charts of the Lyapunov exponents for the system of four dissipatively coupled van der Pol oscillators (1), where $\lambda = 0.1$, $\Delta_2 = 0.03$, $\Delta_3 = 0.1$. The color palette is given and described at the right of the charts. Resonance conditions (2) are indicated by vertical arrows in panel a. Horizontal arrows in panel b correspond to different bifurcations (see the caption for Fig. 2.) CBS is the region of the complete broadband synchronization; PBS is the region of the partial broadband synchronization (see Section 4); AD is the region of the "amplitude death".

[1] Here we consider the five largest Lyapunov exponents. The rest ones are always negative. They do not affect the type of the dynamical regimes. The fifth Lyapunov exponent is also always negative. But it is required to determine the type of the quasi-periodic bifurcation, see Fig. 2 and further discussion.

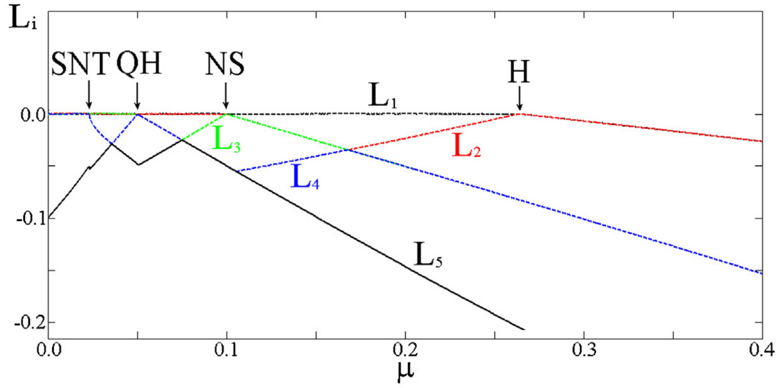

Fig. 2. (Color online.) Graphs of five largest Lyapunov exponents of the system of four dissipatively coupled van der Pol oscillators (1) at the next values of the parameters: $\lambda = 0.1$, $\Delta_1 = 2$, $\Delta_2 = 0.03$, $\Delta_3 = 0.1$. H is the point of the Hopf bifurcation; NS is the point of the Neimark–Sacker bifurcation; QH is the point of the quasi-periodic Hopf bifurcation of two-frequency torus; SNT is the point of the saddle-node bifurcation of three-frequency torus.

Neimark–Sacker bifurcation. The stable two-frequency torus arises at this point.

Then, at the point QH the third exponent L_3 becomes equal to zero too. Thus, before the bifurcation the exponents L_3 and L_4 are equal. This means, that corresponding multipliers are the conjugate complex. At the bifurcation point the exponents L_3 and L_4 are equal to zero. After bifurcation the exponent L_3 is zero, and exponent L_4 is negative again. This is a typical feature of the soft quasi-periodic bifurcation. The three-frequency torus is the result of this bifurcation. This type of bifurcation is called *quasi-periodic Hopf bifurcation* in [6].

At the point SNT dimension of the torus increases once more. However, it is another type of bifurcation. Now the exponent L_4 becomes zero, while exponent L_5 always remains negative. This means, that *the saddle-node bifurcation of tori* takes place [6,7]. The stable and saddle three-frequency tori merge and disappear and the stable four-frequency torus arises. Note, that the last bifurcation has an analog in the phase model, while other bifurcations are not possible in the phase model [13].

3. Increasing of the threshold value of the “amplitude death” region

Let us return to Fig. 1b. Note, that in the case of two or three coupled oscillators the boundary of the “amplitude death” region is given by the condition $\mu = \lambda$ [12,13]. However, in the case of the four coupled oscillators it does not tend to this value as the frequency detuning Δ_1 increases. Actually it is much greater. Thus, the *threshold increase for the “amplitude death” effect* is observed for the four coupled oscillators. From Fig. 1b one can see that the boundary of the “amplitude death” region (point H) corresponds approximately to $\mu \approx 0.26$.

Let us discuss this effect in more detail. In the case of three coupled oscillators synchronization between the first and the second oscillators and between the second and the third oscillators is destroyed as frequency of the second oscillator is increased. As a result, the boundary of the “amplitude death” region is given by $\mu = \lambda$ as $\Delta_1 \to \infty$. It is a condition of the compensation of negative friction by dissipation. In the case of four coupled oscillators the situation is more complicated. Indeed, the third and the fourth oscillators with fixed eigenfrequencies can effectively interact in near-synchronous regimes. As a result, a pair of these oscillators forms much more excited self-oscillating subsystem. So, the threshold value of coupling parameter corresponding to the effect of “amplitude death” can increase.

Let us give some estimates. The truncated equations (Landau–Stuart equations) for the system (1) are[2]:

$$\begin{aligned}
2\dot{a} &= \lambda a - |a|^2 a - \mu(a-b),\\
2\dot{b} &= \lambda b - |b|^2 b + i\Delta_1 b - \mu(2b-a-c),\\
2\dot{c} &= \lambda c - |c|^2 c + i\Delta_2 c - \mu(2c-b-d),\\
2\dot{d} &= \lambda d - |d|^2 d + i\Delta_3 b - \mu(d-c).
\end{aligned} \tag{3}$$

Here $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are slow complex amplitudes of the oscillators (varying slowly in comparison with the basic oscillations with the unit frequency). They are related to the dynamical variables of the system (1) via relations:

$$\begin{aligned}
x &= a(t)e^{it} + a^*(t)e^{-it}, \qquad y = b(t)e^{it} + b^*(t)e^{-it},\\
z &= c(t)e^{it} + c^*(t)e^{-it}, \qquad w = d(t)e^{it} + d^*(t)e^{-it}.
\end{aligned} \tag{4}$$

It is known that linear approximation is enough to determine the “amplitude death” region. We assume that $\Delta_1 \to \infty$, and the eigenfrequencies of the first, the third and the fourth oscillators are small in comparison with Δ_1. Otherwise we assume that $\Delta_2 = \Delta_3 = 0$. Then we rewrite the system (3) as

$$\begin{aligned}
2\dot{a} &= \lambda a - \mu(a-b),\\
2\dot{b} &= \lambda b + i\Delta_1 b - \mu(2b-a-c),\\
2\dot{c} &= \lambda c - \mu(2c-b-d),\\
2\dot{d} &= \lambda d - \mu(d-c).
\end{aligned} \tag{5}$$

We look for the solution of this linear system by the exponential substitution $\exp(\beta t/2)$. Then, we obtain the following equations:

$$\begin{aligned}
(a - \beta + \mu)a &= \mu b,\\
(a - \beta + 2\mu - i\Delta_1)b &= \mu(a+c),\\
(a - \beta + 2\mu)c &= \mu(b+d),\\
(a - \beta + \mu)d &= \mu c.
\end{aligned} \tag{6}$$

From the second equation in (6) we obtain $a/b \to \infty$ or $c/b \to \infty$ if $\Delta_1 \to \infty$. Physically, this corresponds to the fact that either the first or the third oscillator dominates over the second oscillator.

Let us estimate the first ratio. From the first equation in (6) we have

[2] Derivation of these equations is standard for the method of slowly varying amplitudes [1,2]. Therefore we do not reproduce it here.

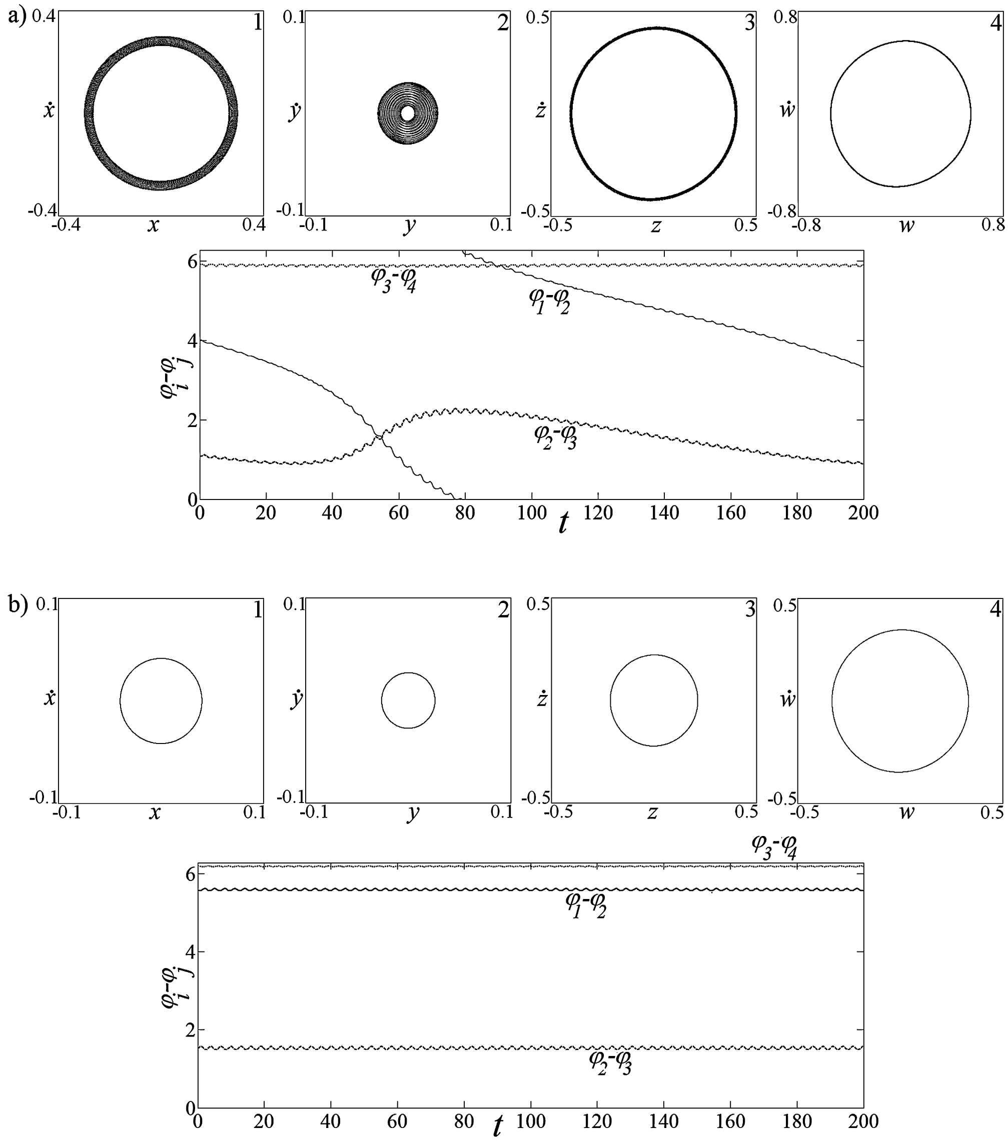

Fig. 3. Phase portraits of the oscillators and phase differences between the oscillators for the system of four dissipatively coupled van der Pol oscillators (1). Values of the parameters are $\lambda = 0.1$, $\Delta_1 = 2$, $\Delta_2 = 0.03$, and $\Delta_3 = 0.1$. a. $\mu = 0.08$. It is the regime of partial broadband synchronization. b. $\mu = 0.2$. It is the regime of complete broadband synchronization. Numbers inside phase portraits correspond to the numbers of the oscillator.

$$a/b = \frac{\mu}{\beta - \lambda + \mu}. \tag{7}$$

Hence we see that $a/b \to \infty$ if $\beta = \lambda - \mu$. The "amplitude death" occurs when $\beta < 0$. So we get $\mu > \lambda$. It is the same condition as in the case of two or three coupled oscillators.

Let us estimate the second ratio. It is associated with the dominance of the third oscillator. From the second and the third equations in (6) we have

$$c/b = \frac{\mu}{\beta - \lambda + 2\mu - \frac{\mu^2}{\beta-\lambda+\mu}}. \tag{8}$$

The condition $c/b \to \infty$ is satisfied when the denominator is zero. This leads to a quadratic equation for β. Then we obtain:

$$\beta = \lambda - \frac{3 \pm \sqrt{5}}{2}\mu. \tag{9}$$

The "amplitude death" regime occurs when $\beta < 0$. Then we have

$$\mu > \frac{2}{3 - \sqrt{5}}\lambda \approx 2.618 \cdot \lambda. \tag{10}$$

Thus, the threshold value of coupling parameter for the "amplitude death" region increases significantly (more than twice) in the case when the third oscillator is dominant and effectively interact with the fourth oscillator. The estimate (10) is quite effective for the case shown in Fig. 1b. Indeed, it gives $\mu \approx 0.26$ for $\lambda = 0.1$. It is very close to the asymptotic boundary of the "amplitude death" region. This is not surprising because the corresponding frequency parameters are small: $\Delta_2 = 0.03$ and $\Delta_3 = 0.1$.

We can still obtain the boundary of the "amplitude death" region for the system (6) in the case when the asymptotic condition $\Delta_1 \to \infty$ is not imposed. Then from Eqs. (6) after some transformations, we obtain

$$z^4 + (2\mu - i\Delta_1)z^3 - \mu(2\mu + i\Delta_1)z^2 - \mu^2(2\mu - i\Delta_1)z + \mu^4 = 0, \tag{11}$$

where $z = \alpha - \lambda + \mu$. The condition of the "amplitude death" of all oscillators is $\mathrm{Re}\,\alpha < 0$. Therefore, the boundary of this region is given by Eq. (11) under the condition

$$\mathrm{Re}\,z = \mu - \lambda. \tag{12}$$

4. Broadband synchronization in a system of four coupled oscillators

Discussed above bifurcations and effects lead to the possibility of broadband synchronization. This regime is observed in the region of large values of frequency detuning Δ_1. There are two types of the regime of the broadband synchronization. First type is a *complete broadband synchronization* when all oscillators are phase locked. In this case the high-dimensional limit cycle occurs. Corresponding synchronization region is located between the lines H and NS. It is marked as CBS in Fig. 1b. Note that for the system of two or three coupled identical oscillators such regime is not observed[3] [13]. For the case of four coupled oscillators it becomes possible due to the increasing of the threshold value of the "amplitude death" region.

Second type is a *partial broadband synchronization*. Corresponding region is marked as PBS in Fig. 1b. In this case only three oscillators are phase locked. The fourth oscillator slightly perturbs their oscillations. So, they are quasi-periodic and correspond to the two-frequency torus. The similar regime is observed in the region between the lines QH and SNT. It corresponds to the three-frequency torus.

Fig. 3 shows the phase portraits of all oscillators and graphs of the phase differences between the oscillators. They are plotted for $\Delta_1 = 2$. Fig. 3a is plotted for the case $\mu = 0.08$. It corresponds to the regime of the partial broadband synchronization PBS (Fig. 1b). One can see that the second oscillator is strongly suppressed (note the different scales on the axes in the phase portraits in Fig. 3). Moreover, its orbit is strongly quasi-periodically perturbed. All other oscillators are excited in almost equal degree. In this case, phase differences of oscillators behave as follows. The phase difference of the third and fourth oscillators $\varphi_3 - \varphi_4$ is constant. The phase difference of the second and the third oscillators $\varphi_2 - \varphi_3$ fluctuates around a constant value. And the phase difference of the first and the second oscillators $\varphi_1 - \varphi_2$ ranges from zero to 2π. Thus, we can say that the second, the third and the fourth oscillators are phase locked.

Fig. 3b is plotted for $\mu = 0.2$. One can see that the quasi-periodic regime is replaced by the periodic regime. In this case both the second and the first oscillators are strongly suppressed. However, despite the high dissipation $\mu > \lambda$, the third and the fourth oscillators are quite strongly excited. Meanwhile, all phase differences $\varphi_1 - \varphi_2$, $\varphi_2 - \varphi_3$ and $\varphi_3 - \varphi_3$ of oscillators are constant.

[3] This regime is possible in the system of two or three dissipatively coupled non-identical oscillators [10,12].

5. Conclusion

Dynamics of the chain of four dissipatively coupled van der Pol oscillators is characterized by a set of specific features. A sequence of bifurcations (Hopf bifurcation, Neimark–Sacker bifurcation, soft quasi-periodic Hopf bifurcation and saddle-node bifurcation) occurs at the large values of the frequency detuning. At the same time, the corresponding bifurcation curves in the parameter plane are the boundaries of the regions of the broadband synchronization of quasi-periodical regimes of different dimensions. The threshold value of the "amplitude death" regime is significantly increased. It can be explained by the interaction between the third and the forth oscillators. Due to the universality of the discussed model the similar behavior can be observed for the various system.

Acknowledgements

The authors thank S.P. Kuznetsov and I.R. Sataev for helpful discussions. The work was supported by the RFBR–DFG grant No. 11-02-91334.

References

[1] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001.
[2] P.S. Landa, Nonlinear Oscillations and Waves in Dynamical Systems, Kluwer Academic Publishers, Dordrecht, 1996.
[3] A.G. Balanov, N.B. Janson, D.E. Postnov, O. Sosnovtseva, Synchronization: From Simple to Complex, Springer, 2009.
[4] L. Glass, M.C. Mackey, From Clocks to Chaos: The Rhythms of Life, Princeton University Press, Princeton, 1988.
[5] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence, Springer, Berlin, 1984.
[6] H. Broer, C. Simó, R. Vitolo, Regul. Chaotic Dyn. 16 (2011) 154–184.
[7] H. Broer, C. Simó, R. Vitolo, Physica D 237 (2008) 1773–1799.
[8] R. Rand, P.J. Holmes, Int. J. Non-Linear Mech. 15 (1980) 387–399.
[9] M. Ivanchenko, G. Osipov, V. Shalfeev, J. Kurths, Physica D 189 (2004) 8–30.
[10] A.P. Kuznetsov, N.V. Stankevich, L.V. Turukina, Physica D 238 (2009) 1203–1215.
[11] D.G. Aronson, G.B. Ermentrout, N. Kopell, Physica D 41 (1990) 403–449.
[12] A.P. Kuznetsov, J.P. Roman, Physica D 238 (2009) 1499–1506.
[13] Y.P. Emelianova, A.P. Kuznetsov, I.R. Sataev, L.V. Turukina, Physica D 244 (2013) 36–49.
[14] A.P. Kuznetsov, I.R. Sataev, L.V. Turukina, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2371–2376.