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Modified Sierpinski Carpet

Galina V. Arzamastseva, Mikhail G. Evtikhov, Feodor V. Lisovsky, Ekaterina G. Mansvetova

Kotelnikov Institute of Radioengineering and Electronics of RAS, Fryazino Branch, http://fire.relarn.ru/

Fryazino 141120, Moscow region, Russian Federation

E-mail: arzamastseva@mail.ru, emg20022002@mail.ru, lisf@df.ru, mansvetova_eg@mail.ru

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Abstract. **The algorithm is described and the properties of a previously unknown modification of the Sierpinsky carpet are studied. An example of proposed algorithm application for the fractal simulation of a really observed domain structure is given. An experimental study of light diffraction in the Fraunhofer zone was performed on computer-generated images of modified Sierpinsky carpets of different generations transferred to a transparent film using a high-resolution imagesetter with a small dot size. The observed diffraction patterns are compared with Fourier images of prefractals pictures approximated by the grid function.**

Keywords: **domain structure, diffraction pattern, Sierpinsky carpet, fractal, Fourier image, Hausdorf dimension**

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Contents

1. INTRODUCTION

Previously, it was reported that a new fractal, which is a Sierpinsky carpet modification [1], was used to simulate a complex domain structure on the surface of uniaxial magnetic single crystalline plates [2]. In this paper, the properties of this fractal are considered in more detail, its Hausdorf dimension is determined, and the results of experiments on the observation of light diffraction by black-and-white bitmap images of different prefractal generations are presented.

The essence of fractal modification is illustrated in **Fig. 1**, where three successive stages of construction of the classic (upper row) and modified (lower row) Sierpinsky carpets are shown. The black color in the drawings is used for displaying fractal elements, and the white color

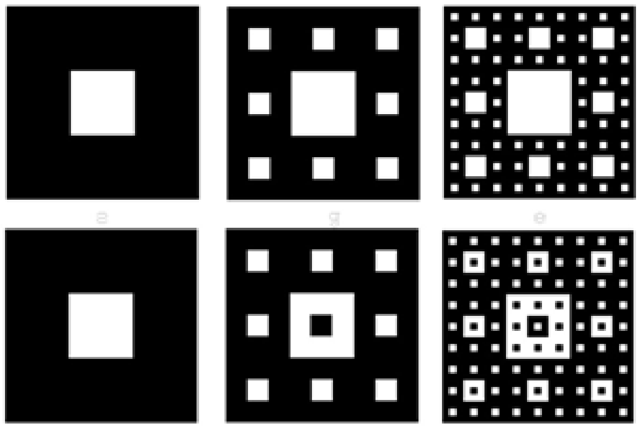

Fig. 1. *Three successive construction stages of the classic (upper row) and modified (lower row) Sierpinsky carpets.*

is used for displaying voids (holes). For a classic carpet, in the first step, the black square with sides of l is divided into nine equal-sized squares with sides of $l/3$, and the middle square is thrown out; then a similar procedure is performed with each of eight black squares bordering the central "white" square, which remains unchanged. Then the process is repeated for sixty-four squares with sides $l/3^2$, etc.

The construction of the modified Sierpinsky carpet also begins with a solitary black square, from which the central square with sides of $l/3$ is removed, but in the second step, in contrast to the classic carpet, a black square with sides of $l/3^2$ is inserted into the center of the figure. The third step is to divide the square into 81 squares in increments $l/3^2$ and then place a square with sides of $l/3^3$ the opposite color (black in white and vice versa) in the center of each of them. This process can be continued indefinitely.

In contrast to the classic Serpinsky carpet, which is a connected topological set, the modified carpet does not have connectivity. Another distinctive feature of this carpet is the lack of strict self-similarity, since the central square for any generation of prefractal differs from all others by inversion of color.

The Hausdorff dimension of a modified Sierpinsky carpet is defined by a formula $D_f = \lim_{n\to\infty} d_n$, where $dn = lnN_n / nln3$, and N_n is the number of black squares with a length of $l/3^n$ in the prefractal of the n-th order. To simplify the calculation (assuming that the value of l is equal to one without limiting generality) let's assume that the central square with a linear size $1/3$, which is an inverse carpet of the n-th order in relation to the original, moves and overlays any of the eight side squares with the same linear size $1/3$. The result is an entirely black square, where the number of black squares with a linear size $1/3^n$ is equal to $3^{2(n-1)}$. Each of the other seven squares with a linear size of $1/3$ is a modified cover (n-1)-th order, that is, the number of black squares with a linear size $1/3^n$ in each of them is equal to N_{n-1}. There is a recurrent relation $N_n = 3^{2(n-1)} + 7N_{n-1}$, which implies that

$$N_n = 3^{2(n-1)} + 7\cdot 3^{2(n-2)} + 7\cdot 3^{2(n-3)} + \ldots +$$
$$+7^{n-2}\cdot 3^2 + 7^{n-1}\cdot 8 =$$
$$= 3^{2(n-1)}\cdot\left[1+\frac{7}{9}+\left(\frac{7}{9}\right)^2+\ldots\left(\frac{7}{9}\right)^{n-2}+8\left(\frac{7}{9}\right)^{n-1}\right]= \quad (1)$$
$$= 3^{2(n-1)}\left[\sum_{k=0}^{k=n}\left(\frac{7}{9}\right)^k+8\left(\frac{7}{9}\right)^n\right].$$

Using the dependence $N_n(n)$ we find that the dimension of the modified Sierpinsky carpet is $D_f = \lim_{n\to\infty}\frac{2(n-2)\ln 3+\ln\left[9/2+8(7/9)^n\right]}{n\ln 3}=2,$ that is, as the order of the prefractal increases, the entire source square is filled in, in contrast to the classic Sierpinsky carpet whose dimension is $ln8/ln3 = 1.89$.

2. THE DIFFRACTION PATTERNS AND THE FOURIER-IMAGES OF THE MODIFIED SIERPINSKI CARPETS

An experimental study of the diffraction of a collimated light beam (with a wavelength of 0.63 microns) in the Fraunhofer zone was performed on computer-generated black-and-white raster images of modified Sierpinski carpets of various generations transferred to a transparent film using a imagesetter with a resolution of 1333 points per centimeter (3386 dpi) and a point size of 7.5 microns. The image of the diffraction pattern on the screen in the diffraction plane was recorded using a digital camera. More detailed methods and features of the described experiments are described in [3].

For numerical determination of Fourier images, black-and-white bitmap images of modified Sierpinsky carpets were approximated by a grid function on a quadrate grid with a number of nodes $n_1\times n_2$, where the values n_1 и n_2 were chosen sufficiently large (up to 4096) to adequately approximate the smallest-size prefractal details (in computer representation) and to enable the study of prefractals with high generation numbers. In our experiments, specific values were chosen so that the parameter p, that is equal to the ratio of the overall linear size of the smallest element of the prefractal to the grid period, was at least 9. For the image digitized in this way, the fast Fourier transform was used to determine the values of the quadrate of the Fourier component modules, i.e., the spectral distribution of the intensity of diffracted radiation in the Fraunhofer zone. To display the intensity of diffraction maxima on the plane, circles with a radius proportional to the intensity (or intensity logarithm) were used [3].

It was found that for modified Sierpinsky carpets, there is a marked difference between their Fourier images, i.e., diffraction patterns calculated from fractal pictures, and those observed in experiments. The central (fractal) parts for all diffraction patterns are almost identical, but the peripheral ("lattice") parts are slightly different. In the experimental picture, there are reflexes from a certain square lattice, which are weakly reflected on the calculated diffractograms. It was found that this discrepancy is due to the difference in the size of isomorphic black and white squares, which was confirmed by observing a model with the image of a fractal on a transparent film under a microscope. When printing with a imagesetter, the laser beam partially illuminates the area outside of the formed ("black") image, as a result of which the black areas (squares) increase in size, and the white (not illuminated) areas, on the contrary, shrink. This difference in size is minimal for large white and black squares and maximal for the smallest squares, where diffraction mainly forms the peripheral part of the diffraction pattern.

The described difference between Fourier images and the experimental diffraction patterns for a modified Sierpinsky carpet depends on the coefficient $c_{wb} = r_w/r_b$, where r_w and r_b, are the linear dimensions of the smallest white and black squares on a transparent film respectively. This is illustrated in **Fig. 2**, which shows experimentally obtained (left) and calculated diffraction patterns at c_{wb} = 1.0 (center) and c_{wb} = 0.64 (right) for a modified carpet of the 6th order. This can be used to reduce the distortion described above by artificially reducing the size of the black squares on the prefractal bitmap images. For the example in Fig. 2 on the right is a calculated diffractogram for a modified Sierpinsky carpet of the 6th order at c_{wb} = 0.64, which corresponds well to the experimental shown on the left, in contrast to the diffractogram in the center for the value c_{wb} = 1.0.

The classic Sierpinski carpet is formed by sublattices consisting only of black squares, so, despite the above-mentioned system error of the imagesetter, the experimental one (on the left in **Fig. 3**) and calculated (on the right in Fig. 3) diffractograms correspond well to each other.

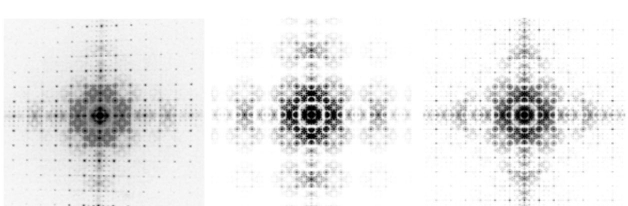

Fig. 2. *Experimentally obtained diffraction pattern (left) and calculated diffraction patterns for (center) and for (right) for a modified Sierpinsky carpet of the 6th order.*

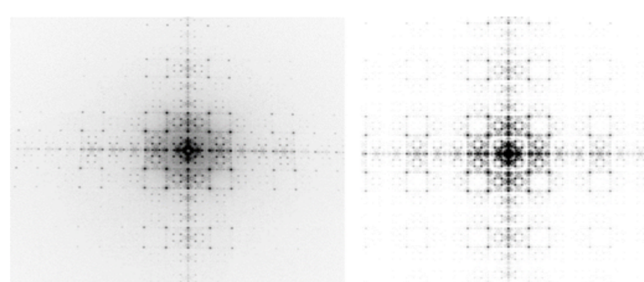

Fig. 3. *Experimentally obtained diffraction pattern (left) and calculated diffraction patterns (right) for a classic Sierpinsky carpet of the 6th order.*

The central part of the diffraction pattern for the modified Sierpinsky carpet has spatial invariance (with a scaling factor equal to 3), as well as for the classical carpet.

Analysis of the diffraction pattern allows us to find the Hausdorff dimension D_f using the circle method (see for example [2]), based on the numerical determination of the average resulting intensity of diffracted radiation $\bar{I}$ in circles with a center at the location of the main diffraction maximum and with a variable radius equal to $r_k = r_0 + k\delta_r$, where r_0 and δ_r is the initial radius and the step of radius change, k = 0,1,2,... and sequential use of the next formula $\bar{I}(r_k) = I_0 \exp(-D_f)$, from which it follows that the fractal dimension is equal to the modulus of the angular coefficient of the line approximating the dependence $\bar{I}(r_k)$ on the double logarithmic scale.

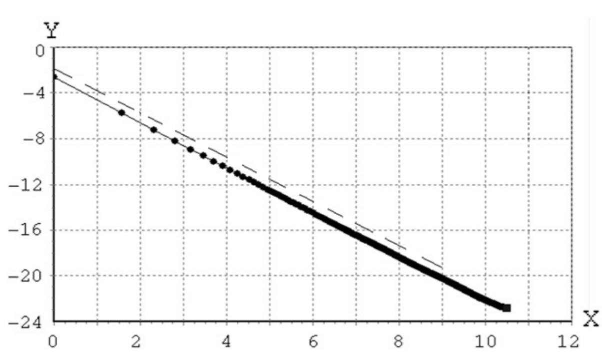

Fig 4. *Dependence for a modified Sierpinsky carpet of the 8th order with the use of normalized variables* $Y = \ln \bar{I}/\ln 2$ *and* $X = lnr_k/ln2$ *on a double logarhyme scale (angular coefficient module of the dashed line equals 1.935).*

For the modified Sierpinsky carpet of the 8th order, the dependence $\bar{I}(r_k)$ for which is on double logarithmic scale using normalized variables $Y = \ln \bar{I}/\ln 2$ and $X = lnr_k/ln2$ is shown in Fig. 4, the value of D_f was 1.936, which is significantly different from 2. The reason is that the value of d_n converges very slowly to the value 2. So, for $n = 8$ the value of D_f equals 1.935, that well corresponds to a certain dimension value (see the dashed line in Fig.4 with module angular coefficient equal to 1.935). Note that the coating method (see e.g. [4]) for this prefractal gives value $D_f = 2$.

3. CONCLUSION

The main results of the work performed are summarized as follows. For the geometric description of a complex domain structure on the surface of the uniaxial magnetic singlecrystal plates, an algorithm based on the use of a previously unknown modification of the Sierpinsky carpet was developed. An example of application of the proposed algorithm for simulation a real observed domain structure is given. An experimental study of light diffraction in the Fraunhofer zone was performed on computer generated images of modified Sierpinsky carpets of different generations transferred to a transparent film using a high-resolution imagesetter with a small dot size. The observed diffraction patterns were compared with Fourier images of prefractals pictures approximated by the grid function. The difference between experimental and calculated diffraction patterns for the modified Sierpinsky carpet was found to be an artifact, and the reason for this difference was revealed.

The work was carried out at the expense of budget financing within the framework of the state assignment.

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