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Nonlinear Elastic Waves in Magnetically Ordered Crystals in the Vicinity of Orientational Phase Transitions

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Abstract—A phase diagram of the dynamic magnetoelastic states of an easy-plane antiferromagnet is constructed. A dispersion relation is obtained for nonlinear magnetoelastic eigenwaves. It is shown that, at the point of the orientational phase transition, the dispersion of coupled spin and elastic waves depends only on wave amplitudes and parameters of magnetoelastic coupling.

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INTRODUCTION

Coupling of electric, magnetic, and elastic subsystems of material media leads to various physical effects. In particular, the spectra of magnetically ordered solids exhibit elementary excitations of coupled magnetoelastic waves (MEWs) the propagation of which leads to variations in both strain and magnetization. The maximum effect is observed in the vicinity of the points of phase matching where frequencies ω and wave numbers k of the spin and elastic waves coincide in the absence of coupling of the subsystems. A known example is the hybridization of spin and elastic waves in ferro- and ferrimagnets that has been theoretically analyzed in [1] (see also [2, 3]). The first experiments that prove the theory have been reported in [4–6].

Effect of magnetoelastic interaction on wave processes is more complicated in antiferromagnets (AFMs). The lower branch of the dispersion curves of spin waves exhibits anomalous behavior at $k \to 0$ in AFMs with easy-plane anisotropy [7–9] due to spontaneous magnetoelastic strains in the ground state [10]. Such an effect was called magnetoelastic gap in [11]. Easy-plane AFMs with relatively weak ferromagnetism in the neighborhood of the orientational phase transition (OPT) exhibit strong dependence of the speed of sound on magnetic field strength in the basal plane [12–14]. Effect of pressure and magnetic field on the propagation of linear eigen MEWs in uniaxial AFMs has been reported in [15–17]. In the vicinity of the OPT, such waves that are neither harmonic nor linear even at small amplitudes owing to orientational instability of the magnetization vectors of sublattices exhibit strong dispersion (nonlinear eigen MEWs: solitary, cnoidal, and helical [18–24]). Effects of magnetoelastic coupling on shock waves and anharmonicity of acoustic waves have been studied in [25] and [26, 27], respectively.

In the vicinity of OPT, a spontaneous violation of symmetry leads to several nontrivial effects. The theoretical calculations of [28, 29] and the supporting experimental results of [30] show that a low-frequency elastic wave in an easy-plane ferromagnetic material may cause transitions from one state to another in the regions of tension and compression with formation of a traveling domain structure consisting of different phases. A high-intensity elastic wave generates spatiotemporal periodicity in a medium and provides conditions for stable and unstable parametric interactions between spatiotemporal harmonics of spin waves [31, 32]. Nonlinear MEWs of different types in the vicinity of OPT can be considered as dynamic states with a certain symmetry that is modified when the type of wave changes. Modification of such states in the presence of variable elastic stress is similar to modification of static states in the presence of static stress (i.e., represents a phase transition).

In this work, we study the phase diagram of dynamic magnetoelastic states and specific features of solitary and coupled MEWs in the vicinity of OPT in an easy-plane antiferromagnet.

1. BASIC EQUATIONS

We solve the problem in the framework of the theory of classic fields [33, 34] using the Lagrangian of an elastically stressed multisublattice magnetic material

$$L = T - U, \tag{1}$$

where

$$T = -\sum_n \frac{M_0^{(n)}}{g^{(n)}} \frac{\partial \phi^{(n)}}{\partial t} \cos\theta^{(n)} + \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2$$

is the kinetic potential [35], $0 \le \theta^{(n)} \le \pi$ and $0 \le \phi^{(n)} \le 2\pi$ are the polar and azimuth angles of the magnetization vector of the n th sublattice

$$\vec{M}^{(n)} = M_0^{(n)}\left\{\sin\theta^{(n)}\cos\phi^{(n)}, \sin\theta^{(n)}\sin\phi^{(n)}, \cos\theta^{(n)}\right\},$$

$M_0^{(n)}$ and $g^{(n)}$ are the saturation magnetization and gyromagnetic coefficient of the n th sublattice, ρ is the material density, $U = U\left(M_i^{(n)}, \partial M_i^{(n)}/\partial x_n, u_{ij}\right)$ is the potential energy, $u_{ij} = \left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)/2$ are the components of the strain tensor, and $\vec{u}$ is the vector of elastic displacements. The Lagrange equation is written as

$$\frac{\partial L}{\partial q_n} = \frac{\partial}{\partial x^{(\nu)}} \frac{\partial L}{\partial\left(\partial q_n/\partial x^{(\nu)}\right)}, \quad (2)$$

$q_n = \left\{\theta^{(n)}, \phi^{(n)}, u_x, u_y, u_z\right\}$, $x^{(\nu)} = \left\{x^{(0)} = t, x^{(1)} = x, x^{(2)} = y, x^{(3)} = z\right\}$, the energy–momentum four-vector of the field is given by $P^{(\nu)} = \int T^{(\nu 0)}\left(x^{(o)}, x\right) d^3x$, where the field tensor is

$$T^{(\nu\mu)} = \frac{\partial L}{\partial\left(\partial q_n/\partial x^{(\mu)}\right)} \frac{\partial q_n}{\partial x^{(\nu)}} - g^{(\nu\mu)} L;$$

$g^{(\nu\mu)}$ is 0 at $\nu \ne \mu$, +1 at $\nu = \mu = 0$, and −1 at $\nu = \mu = 1,2,3$. Invariance of $P^{(\nu)}$ leads to the equation of continuity

$$\frac{\partial T^{(\nu\mu)}}{\partial x^{(\mu)}} = \frac{\partial T^{(\nu 0)}}{\partial t} + \frac{\partial T^{(\nu 1)}}{\partial x} + \frac{\partial T^{(\nu 2)}}{\partial y} + \frac{\partial T^{(\nu 3)}}{\partial z}.$$

Field energy density $T^{(00)}$ and fluence $T^{(0\mu)}$ are independent of gyroscopic terms and given by

$$T^{(00)} = \frac{\partial L}{\partial\left(\partial q_n/\partial t\right)} \frac{\partial q_n}{\partial t} - L \equiv H = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + U;$$

$$T^{(0\mu)} = \frac{\partial L}{\partial\left(\partial q_n/\partial x^{(\mu)}\right)} \frac{\partial q_n}{\partial t},$$

and the density of the $x^{(\nu)}$-component of the field momentum is represented as

$$T^{(\nu 0)} = \frac{\partial L}{\partial\left(\partial q_n/\partial t\right)} \frac{\partial q_n}{\partial x^{(\nu)}}$$

$$= -\sum_n \frac{M_0^{(n)}}{g^{(n)}} \cos\theta^{(n)} \frac{\partial \phi^{(n)}}{\partial x^{(\nu)}} + \rho \frac{\partial \vec{u}}{\partial t} \frac{\partial \vec{u}}{\partial x^{(\nu)}}.$$

With allowance for dissipation, the Lagrange equations are written as

$$\frac{d}{dt} \frac{\partial L}{\partial\left(\partial q_n/\partial t\right)} + \frac{\delta L_d}{\delta\left(\partial q_n/\partial t\right)} - \frac{\delta L}{\delta q_n} = 0, \quad (3)$$

where the dissipative function is represented as

$$L_d = \sum_n \frac{1}{2} \frac{M_0^{(n)}}{g^{(n)}} r^{(n)} \left(\left(\frac{\partial \theta^{(n)}}{\partial t}\right)^2 + \left(\frac{\partial \phi^{(n)}}{\partial t}\right)^2 \sin^2\theta^{(n)}\right) + \frac{1}{2}\eta_{ijkn} \frac{\partial u_{ij}}{\partial t} \frac{\partial u_{kn}}{\partial t},$$

$r^{(n)}$ are the dissipative coefficients of magnetization, and η_{ijkn} are the components of elastic dissipative tensor,

$$\frac{\delta}{\delta q_n} \equiv \frac{\partial}{\partial q_n} - \frac{\partial}{\partial x_n} \frac{\partial}{\partial\left(\partial q_n/\partial x_n\right)}.$$

The following equations of motion are obtained from expressions (1) and (2):

$$\frac{M_0^{(n)}}{g^{(n)}}\left(-\frac{\partial \phi^{(n)}}{\partial t} \sin\theta^{(n)} + r^{(n)} \frac{\partial \theta^{(n)}}{\partial t}\right) = \frac{\partial}{\partial x_n} \frac{\partial U}{\partial\left(\partial \theta^{(n)}/\partial x_n\right)} - \frac{\partial U}{\partial \theta^{(n)}},$$

$$\frac{M_0^{(n)}}{g^{(n)}} \sin\theta^{(n)}\left(\frac{\partial \theta^{(n)}}{\partial t} + r^{(n)} \frac{\partial \phi^{(n)}}{\partial t} \sin\theta^{(n)}\right) = \frac{\partial}{\partial x_n} \frac{\partial U}{\partial\left(\partial \phi^{(n)}/\partial x_n\right)} - \frac{\partial U}{\partial \phi^{(n)}}, \quad (4)$$

$$\rho \frac{\partial^2 u_n}{\partial t^2} + \frac{1}{2}(1 + \delta_{mn}) \frac{\partial \sigma_{nm}^{(d)}}{\partial x_m} = -\frac{1}{2}(1 + \delta_{mn}) \frac{\partial \sigma_{nm}}{\partial x_m},$$

where $\sigma_{mn} = -\partial U/\partial u_{mn}$ and $\sigma_{nm}^{(d)} = -\eta_{nmkl}\left(\partial u_{kl}/\partial t\right)$ are the elastic and viscous stresses.

For the system under study, the potential energy contains contributions of magnetic subsystem U_m, elastic subsystem U_u, interaction of the subsystems U_{mu}, and interaction of resulting magnetization $\vec{M}_s$ with external magnetic field $\vec{H}_0$ and demagnetizing field $\vec{H}_d$, so that $U = U_m + U_u + U_{mu} - \vec{M}_s\left(\vec{H}_0 + \vec{H}_d/2\right)$. The contribution of the magnetic subsystem is

$$U_m \equiv U_m\left(M_i^{(n)}, \partial M_j^{(n)}/\partial x_n\right) = U_{mn} + U_{mh},$$

where

$$U_{mn} = \sum_{n,m=1}^{n,m=N} a_{ij}^{(nm)} \frac{\partial \vec{M}^{(n)}}{\partial x_i} \frac{\partial \vec{M}^{(m)}}{\partial x_j}$$

is the potential of inhomogeneous exchange interaction,

$$U_{mh} = \sum_{n,m=1}^{n,m=N} b_{ij}^{(nm)} M_i^{(n)} M_j^{(m)} + \sum_{n,m,p,q=1}^{n,m,p,q=N} b_{ijkl}^{(nmpq)} M_i^{(n)} M_j^{(m)} M_k^{(p)} M_l^{(q)}$$

is the homogeneous potential in which the terms that represent scalar products determine the energy of homogeneous exchange interaction and the remaining terms determine the anisotropy energy, $a_{ij}^{(nm)}$, $b_{ij}^{(nm)}$, and $b_{ijkl}^{(nmpq)}$ are the phenomenological constants, $U_u = (1/2)\lambda_{ijkl} u_{ij} u_{kl}$ is the potential of elastic subsystem,

$$U_{mu} = \sum_{n,m=1}^{n,m=N} c_{ijkl}^{(nm)} M_i^{(n)} M_j^{(m)} u_{kl}$$

is the potential of magnetoelastic interaction,

$$\vec{M}_s = \sum_{n=1}^{n=N} \vec{M}^{(n)}$$

is the resulting magnetization, and $\vec{H}_d$ is the demagnetizing field that satisfies the equations $\operatorname{div}\vec{H}_d = -4\pi \operatorname{div} \vec{M}_s$ and $\operatorname{rot}\vec{H}_d = 0$. Dynamic magnetoelastic states are determined by Eqs. (2), and homogeneous and inhomogeneous static states are determined by the equations

$$\delta U/\delta\theta^{(n)} = 0; \quad \delta U/\delta\phi^{(n)} = 0; \quad \delta\sigma_{nm}/\delta x_m = 0. \quad (5)$$

Using Eqs. (4), we represent the equations of motion for magnetization vectors of the sublattices as

$$\begin{gathered}
\omega_M^{-1}\left[-\frac{\partial\phi^{(\pm)}}{\partial t}\sin\theta^{(+)}\cos\theta^{(-)} - \frac{\partial\phi^{(\mp)}}{\partial t}\sin\theta^{(-)}\cos\theta^{(+)} + r\frac{\partial\theta^{(\mp)}}{\partial t}\right] \\
= \frac{\partial}{\partial x_n}\frac{\partial \underline{u}}{\partial\left(\partial\theta^{(\pm)}/\partial x_n\right)} - \frac{\partial \underline{u}}{\partial\theta^{(\pm)}}, \\
\omega_M^{-1}\left[\frac{\partial\theta^{(\pm)}}{\partial t}\sin\theta^{(+)}\cos\theta^{(-)} - \frac{\partial\theta^{(\mp)}}{\partial t}\sin\theta^{(-)}\cos\theta^{(+)} \right. \\
+ r\frac{\partial\phi^{(\mp)}}{\partial t}\left(\sin^2\theta^{(+)}\cos^2\theta^{(-)} + \sin^2\theta^{(-)}\cos^2\theta^{(+)}\right) \\
\left. + 2r\frac{\partial\phi^{(\pm)}}{\partial t}\sin\theta^{(+)}\cos\theta^{(-)}\sin\theta^{(-)}\cos\theta^{(+)}\right] \\
= \frac{\partial}{\partial x_n}\frac{\partial \underline{u}}{\partial\left(\partial\phi^{(\pm)}/\partial x_n\right)} - \frac{\partial \underline{u}}{\partial\phi^{(\pm)}},
\end{gathered} \quad (6)$$

where $\theta^{(\pm)} = \left(\theta^{(1)} \pm \theta^{(2)} + \pi\right)/2$, $\theta^{(1)} = \theta^{(+)} + \theta^{(-)}$, $\theta^{(2)} = \theta^{(+)} - \theta^{(-)} + \pi$, $\phi^{(\pm)} = \left(\varphi^{(1)} \pm \phi^{(2)} + \pi\right)/2$, $\phi^{(1)} = \phi^{(+)} + \phi^{(-)}$, $\phi^{(2)} = \phi^{(+)} - \varphi^{(-)} + \pi$, $\omega_M = 2M_0 g$, $M_0 = |\vec{M}^{(1)}| = |\vec{M}^{(2)}|$, $g = g^{(1)} = g^{(2)}$, $r = r^{(1)} = r^{(2)}$, and $\underline{u} = U/(2M_0)^2$ is the normalized homogeneous potential. Angles $\theta^{(-)}$ and $\phi^{(-)}$ determine the degree of noncollinearity of the magnetization vectors of sublattices.

2. GROUND STATE

We consider a tetragonal antiferromagnetic material in the presence of external uniaxial elastic stress exerted along the edges of a unit cell. We restrict consideration to a system that is far from the spin-flop transition when $|\vec{m}| \ll |\vec{l}|$, where $\vec{m} = \left(\vec{M}^{(1)} + \vec{M}^{(2)}\right)/(2M_0)$ and $\vec{l} = \left(\vec{M}^{(1)} - \vec{M}^{(2)}\right)/(2M_0)$ are the normalized vectors of ferromagnetism and antiferromagnetism, respectively. In such an approximation, the terms of the normalized potential $\underline{u} = \underline{u}_m + \underline{u}_{mu} + \underline{u}_u$ can be represented as

$$\begin{gathered}
\underline{u}_m = a_0 m^2 + a_1 m_z^2 + k_1 l_z^2 + k_2\left(l_x^4 + l_y^4 + l_z^4\right) \\
+ k_4\left(l_x^2 l_y^2 + \left(l_x^2 + l_y^2\right) l_z^2\right), \\
\underline{u}_{mu} = b_1\left(l_x^2 u_{xx} + l_y^2 u_{yy}\right) + b_2 l_z^2\left(u_{xx} + u_{yy}\right) \\
+ b_3 l_z^2 u_{zz} + b_4\left(l_x l_z u_{xz} + l_y l_z u_{yz}\right) + b_6 l_x l_y u_{xy}, \\
\underline{u}_u = (1/2)\left[\lambda_{11}\left(u_{xx}^2 + u_{yy}^2\right) + \lambda_{33} u_{zz}^2\right] \\
+ \lambda_{12} u_{xx} u_{yy} + \lambda_{13}\left(u_{xx} + u_{yy}\right) u_{zz} \\
+ 2\lambda_{44}\left(u_{xz}^2 + u_{yz}^2\right) + 2\lambda_{66} u_{xy}^2,
\end{gathered} \quad (7)$$

where a_0 and a_1 are the exchange constants; k_1, k_2, and k_4 are the constants of magnetocrystalline anisotropy, b_i are the magnetoelastic constants, and λ_{ik} are the elasticity coefficients.

In the absence of stress in the AFM at $a_0, a_1 > 0$, the ground static state corresponds to the absence of resulting magnetization. At relatively large constant $k_1 > 0$, we obtain the easy-plane state. At $k_4 - 2k_2 > 0$, we have one of two degenerate collinear plane states: $l_y = 0, l_x^2 = 1$ or $l_y^2 = 1$, $l_x = 0$. External elastic stresses $\sigma_{ij}^{(0)}$ normalized by $(2M_0)^2$ cause equilibrium strain $u_{ij}^{(0)}$ that can be represented in the following way using equations $\sigma_{ij}^{(0)} = \partial\underline{u}/\partial u_{ij}$:

$$\begin{gathered}
u_{xx}^{(0)} = -b_{11}^*\left(l_x^{(0)}\right)^2 - b_{12}^*\left(l_y^{(0)}\right)^2 - b_{13}^*\left(l_z^{(0)}\right)^2 \\
+ \lambda_{10}\sigma_{xx}^{(0)} + \lambda_{20}\sigma_{yy}^{(0)} + \lambda_{30}\sigma_{zz}^{(0)}, \\
u_{yy}^{(0)} = -b_{12}^*\left(l_x^{(0)}\right)^2 - b_{11}^*\left(l_y^{(0)}\right)^2 - b_{13}^*\left(l_z^{(0)}\right)^2 \\
+ \lambda_{20}\sigma_{xx}^{(0)} + \lambda_{10}\sigma_{yy}^{(0)} + \lambda_{30}\sigma_{zz}^{(0)}, \\
u_{zz}^{(0)} = -b_{31}^*\left(l_x^{(0)}\right)^2 - b_{31}^*\left(l_y^{(0)}\right)^2 - b_{33}^*\left(l_z^{(0)}\right)^2
\end{gathered}$$

$$+\lambda_{30}\sigma_{xx}^{(0)}+\lambda_{30}\sigma_{yy}^{(0)}+\lambda_{40}\sigma_{zz}^{(0)},$$
$$u_{xy}^{(0)}=-b_{66}^{*}l_x^{(0)}l_y^{(0)},\quad u_{xz}^{(0)}=-b_{44}^{*}l_x^{(0)}l_z^{(0)},$$
$$u_{yz}^{(0)}=-b_{44}^{*}l_y^{(0)}l_z^{(0)};\quad b_{11}^{*}=\lambda_{10}b_1,\quad b_{12}^{*}=\lambda_{20}b_1, \tag{8}$$
$$b_{13}^{*}=(\lambda_{10}+\lambda_{20})b_2+\lambda_{30}b_3,\quad b_{31}^{*}=\lambda_{30}b_1,$$
$$b_{33}^{*}=2\lambda_{30}b_2+\lambda_{40}b_3,$$
$$\lambda_{i0}=\lambda_i/\lambda_0\,(i=1\text{–}4),\quad \lambda_1=\lambda_{11}\lambda_{33}-\lambda_{13}^2,$$
$$\lambda_2=\lambda_{13}^2-\lambda_{12}\lambda_{33},\quad \lambda_3=(\lambda_{12}-\lambda_{11})\lambda_{13},$$
$$\lambda_4=\left(\lambda_{11}^2-\lambda_{12}^2\right),\quad \lambda_0=\left(\lambda_{11}^2-\lambda_{12}^2\right)\lambda_{33}$$
$$+2\lambda_{13}^2(\lambda_{12}-\lambda_{11}),\quad b_{66}^{*}=b_6/(2\lambda_{66}),$$
$$b_{44}^{*}=b_4/(2\lambda_{44}).$$

Substituting expressions for $u_{ij}^{(0)}$ from (8) to (7), we find the equilibrium potential

$$\underline{u}_0=a_0\left(m^{(0)}\right)^2+a_1\left(m_z^{(0)}\right)^2+k_1'\left(l_z^{(0)}\right)^2$$
$$+k_2'\left(l_z^{(0)}\right)^4+k_2''\left(\left(l_x^{(0)}\right)^2+\left(l_y^{(0)}\right)^2\right)^2$$
$$+k_{24}\left(l_x^{(0)}\right)^2\left(l_y^{(0)}\right)^2+k_4'\left(\left(l_x^{(0)}\right)^2+\left(l_y^{(0)}\right)^2\right)\left(l_z^{(0)}\right)^2 \tag{9}$$
$$+\sigma_{11}\left(l_x^{(0)}\right)^2+\sigma_{22}\left(l_y^{(0)}\right)^2,$$

where $k_1'=k_1+\sigma_{33}$, $k_2'=k_2-2b_2b_{13}^{*}-b_3b_{33}^{*}$, $k_2''=k_2-b_1b_{11}^{*}$, $k_{24}=k_4-2k_2+2b_1\left(b_{11}^{*}-b_{22}^{*}\right)-2b_6b_{66}^{*}$, and $k_4'=k_4-b_2\left(b_{11}^{*}+b_{12}^{*}\right)-b_1b_{31}^{*}-2b_4b_{44}^{*}$ are the anisotropy constants renormalized by magnetostriction and

$$\sigma_{11}=b_1\left(\lambda_{10}\sigma_{xx}^{(0)}+\lambda_{20}\sigma_{yy}^{(0)}+\lambda_{30}\sigma_{zz}^{(0)}\right),$$
$$\sigma_{22}=b_1\left(\lambda_{20}\sigma_{xx}^{(0)}+\lambda_{10}\sigma_{yy}^{(0)}+\lambda_{30}\sigma_{zz}^{(0)}\right),$$
$$\sigma_{33}=[b_2(\lambda_{10}+\lambda_{20})+b_3\lambda_{30}]\left(\sigma_{xx}^{(0)}+\sigma_{yy}^{(0)}\right)$$
$$+(2b_2\lambda_{30}+b_3\lambda_{40})\sigma_{zz}^{(0)}$$

are the effective stresses. Magnetoelastic coupling leads to renormalization of magnetic constants, and the external stress leads to additional contribution to the energy of the plane uniaxial anisotropy. When such a contribution is sufficiently large and $k_1'>0$, the ground state is plane and, in addition, collinear in the absence of external stress at $k_{24}>0$. External stresses may cause OPT with respect to polar or azimuth angles.

Using the formulas for components $\vec{m}$ and $\vec{l}$ on spherical coordinates, we find the conditions for existence of equilibrium states:

$$f_{\theta1}^{(-)}\sin 2\theta_0^{(-)}+(1/2)f_{\theta2}^{(-)}\cos 2\theta_0^{(-)}$$
$$\times\sin 2\theta_0^{(+)}\sin 2\phi_0^{(-)}=0, \tag{10a}$$
$$f_{\phi1}^{(-)}\sin 2\phi_0^{(-)}$$
$$+f_{\phi2}^{(-)}\sin 2\theta_0^{(-)}\sin 2\theta_0^{(+)}\cos 2\phi_0^{(-)}=0, \tag{10b}$$
$$f_{\theta1}^{(+)}\sin 2\theta_0^{(+)}+f_{\theta2}^{(+)}\cos 2\theta_0^{+}\sin 2\theta_0^{(-)}$$
$$\times\sin 2\phi_0^{+}\sin 2\phi_0^{(-)}=0, \tag{10c}$$
$$f_{\phi1}^{(+)}\cos 2\phi_0^{(+)}+f_{\phi2}^{(+)}\sin 2\phi_0^{(+)}=0, \tag{10d}$$

where $f_{\theta1,2}^{(-)}$, $f_{\theta1,2}^{(+)}$, $f_{\phi1,2}^{(-)}$, $f_{\phi1,2}^{(+)}$, are the functions that determine angular (with respect to polar angle) phases (explicit cumbersome expressions are omitted).

The analysis of Eqs. (10a)–(10d) shows that one of the following phases $S^{(s)}$ is allowed for the equilibrium static state of AFM. In the collinear (with respect to vectors $\vec{M}^{(1)}$ and $\vec{M}^{(2)}$) phase $S_1^{(s)}$ for which $\theta_0^{(-)}=\phi_0^{(-)}=0$, Eqs. (10a) and (10b) are automatically satisfied and the orientation of the antiferromagnetic vector is determined by the following expressions:

$$\sin 2\theta_0^{(+)}f_0=0;\ \left(\sin\theta_0^{(+)}\right)^2\sin 2\phi_0^{(+)}$$
$$\times\left[k_{24}\left(\sin\theta_0^{(+)}\right)^2\cos 2\phi_0^{(+)}+\sigma_{22}-\sigma_{11}\right]=0,$$

where $f_0=f_0\left(\theta_0^{(+)},\phi_0^{(+)}\right)$ is the function of the polar and azimuth angles of vector $\vec{l}$.

Note also possibility of axial phase $S_{1.1}^{(s)}$ for which $\theta_0^{(+)}=0$ and plane phases $S_{1.2}^{(s)}$ for which $\theta_0^{(+)}=\pi/2$ in the three modifications: two collinear $S_{1.2.1}^{(s)}$ and $S_{1.2.2}^{(s)}$ ($\phi_0^{(+)}=0$ and $\phi_0^{(+)}=\pi/2$, respectively) and angular (with respect to azimuth angle) $S_{1.2.3}^{(s)}$ ($\cos 2\phi_0^{(+)}=(\sigma_{11}-\sigma_{22})/k_{24}$).

Angular phase with respect to the polar angle $S_{1.3}^{(s)}$ is also implemented in the three modifications: two collinear $S_{1.3.1}^{(s)}$ and $S_{1.3.2}^{(s)}$ for which

$$\phi_0^{(+)}=0;\quad \cos^2\theta_0^{(+)}=\left(\sigma_{11}-k_1'\right)/(2k_{24}),$$

and

$$\phi_0^{(+)}=\pi/2;$$
$$\cos^2\theta_0^{(+)}=\left(\sigma_{22}-k_1'-k_4'+2k_2''\right)/\left(2k_{24}'\right),$$

and angular (with respect to azimuth) $S_{1.3.3}^{(s)}$ for which

$$\cos 2\phi_0^{(+)}(\sigma_{11}-\sigma_{22})$$
$$\times\left(4k_{24}'+k_{24}\right)/\left[k_{24}\left(2k_1'+4k_{24}'-\sigma_{11}-\sigma_{22}\right)\right];$$

$$\sin^2\theta_0^{(+)} = \left(2k_1' + 4k_{24}' - \sigma_{11} - \sigma_{22}\right)/\left(4k_{24}' + k_{24}\right),$$

and $k_{24}' = k_2' + k_2'' - k_4'$.

3. MAGNETOELASTIC EIGENWAVES IN THE VICINITY OF OPT

We consider evolution of nonlinear waves in the vicinity of OPT between plane collinear phase $S_{12}^{(s)}$ and angular phase $S_{123}^{(s)}$ at point $\sigma_{11} - \sigma_{22} = k_{24}$. At a relatively large axial anisotropy constant and low attenuation, the following relationship follows from the first equation of system (6):

$$\omega_M^{-1}\left(\partial\phi^{(+)}/\partial t\right) = -k_1 \sin 2\theta^{(+)}, \quad (11)$$

and the second equation of system (6) and elasticity equations depending on variables (x, y) are represented as

$$\begin{aligned}
&k_1^{-1}\omega_M^{-2}\left(\partial^2\phi^{(+)}/\partial t^2\right) = a_{11}\left(\partial^2\phi^{(+)}/\partial x^2\right) \\
&\quad - \left[(1/2)k_4 - k_2\right]\sin 4\phi^{(+)} \\
&\quad + b_1\left(u_{xx}^{(0)} - u_{yy}^{(0)} + \partial u_x/\partial x\right)\sin 2\phi^{(+)} \\
&\quad - (1/2)b_6 \cos 2\phi^{(+)}\,\partial u_y/\partial x; \\
&\rho\left(\partial^2 u_x/\partial t^2\right) = \lambda_{11}\left(\partial^2 u_x/\partial x^2\right) \\
&\quad + (1/2)b_1\,\partial\left(\cos 2\phi^{(+)}\right)/\partial x; \\
&\rho\left(\partial^2 u_y/\partial t^2\right) = \lambda_{66}\left(\partial^2 u_y/\partial x^2\right) \\
&\quad + (1/4)b_6\,\partial\left(\sin 2\phi^{(+)}\right)/\partial x.
\end{aligned} \quad (12)$$

Expressions (12) show that elastic strain for waves with velocity v that are represented as $\left\{\phi^{(+)}, u_x, u_y\right\} \propto f(\xi)$, where $\xi = (x - vt)$, is given by

$$u_{xx} = b_{1v}\left(\cos 2\phi^{(+)} - 1\right), \quad u_{yx} = b_{6v}\sin 2\phi^{(+)}, \quad (13)$$

where $b_{1v} = b_1/2\rho\left(v^2 - v_l^2\right)$, $b_{6v} = b_6/4\rho\left(v^2 - v_{t1}^2\right)$ are the dynamic magnetoelastic coefficients and $v_l^2 = \lambda_{11}/\rho$ and $v_{t1}^2 = \lambda_{66}/\rho$ are the velocities of elastic longitudinal y-polarized transverse waves. The Newton equation for a spin quasiparticle is written as

$$m^*\partial^2\phi^{(+)}/\partial\xi^2 = -\partial\underline{u}^{\text{эф}}/\partial\phi^{(+)}, \quad (14)$$

where $m^* = a_{11}\left(v^2/s^2 - 1\right)$ is the effective mass that is positive at $v^2 > s^2$ and negative at $v^2 < s^2$, where $s = {}_1\omega_M\sqrt{a_{11}k_1}$ is the parameter with the dimension of velocity and $\underline{u}^{(\text{eff})} = \sin^2\phi^{(+)}\left(k_{u2}\cos^2\phi^{(+)} - k_{u1}\right)$ is the effective potential taking into account tetragonal anisotropy with constant that is renormalized by dynamic magnetoelastic interaction $k_{u2} = k_4 - 2k_2 - b_1 b_{1v} + 1/2\, b_6 b_{6v}$ and uniaxial anisotropy with constant determined by static and dynamic magnetoelastic interaction $k_{u1} = b_1\left(u_{xx}^{(0)} - u_{yy}^{(0)} - b_{1v}\right)$. In accordance with expression (14), the law of energy conservation for a quasiparticle can be represented as $w + \underline{u}^{\text{eff}} = \underline{e}$, where $w = (1/2)m^*\left(\partial\phi^{(+)}/\partial\xi\right)^2$ is the kinetic energy and integration constant $\underline{e}$ serves as the total energy of quasiparticle. An implicit solution to Eq. (14) is given by

$$\int\left(\underline{e} - \underline{u}^{\text{eff}}\right)^{-1/2} d\phi^{(+)} = \pm\left(2/m^*\right)^{1/2}(\xi - \xi_0). \quad (15)$$

Type and behavior of MEW depend on function $\underline{u}^{\text{eff}}\left(\phi^{(+)}\right)$ the extremum values of which on interval $-1 \le k_{u1}/k_{u2} \le 1$ are reached at $\sin\phi_{m1}^{(+)} = 0$, $\cos\phi_{m2}^{(+)} = 0$ and $\cos 2\phi_{m3}^{(+)} = k_{u1}/k_{u2}$:

$$\underline{u}_1^{\text{eff}} \equiv \underline{u}^{\text{eff}}\left(\phi_{m1}^{(+)}\right) = 0,$$

$$\underline{u}_2^{\text{eff}} \equiv \underline{u}^{\text{eff}}\left(\phi_{m2}^{(+)}\right) = -k_{u1},$$

$$\underline{u}_3^{\text{eff}} \equiv \underline{u}^{\text{eff}}\left(\phi_{m3}^{(+)}\right) = \left(k_{u2} - k_{u1}\right)^2/4\,k_{u2}.$$

For $k_{u2} > 0$, the potential reaches an absolute minimum at points $\phi_{m1}^{(+)}$ when $k_{u1} < 0$ and at points $\phi_{m2}^{(+)}$ when $k_{u1} > 0$. The maxima are located at points $\phi_{m2}^{(+)}$ at $k_{u1} < -k_{u2}$, points $\phi_{m3}^{(+)}$ at $k_{u2} < k_{u1} < -k_{u2}$, and points $\phi_{m1}^{(+)}$ at $k_{u1} > k_{u2}$. When $k_{u2} < 0$, minimum of $\underline{u}^{\text{eff}}$ is reached at points $\phi_{m1}^{(+)}$ at $k_{u1} < k_{u2}$, points $\phi_{m3}^{(+)}$ for interval $k_{u2} < k_{u1} < -k_{u2}$, and points $\phi_{m2}^{(+)}$ at $k_{u1} < -k_{u2}$. The period of the potential is π, and the period decreases by a factor of 2 at point $k_{u1} = 0$.

Figure 1 shows dependence $\underline{u}^{\text{eff}}\left(\phi^{(+)}\right)$ for $k_{u2} > 0$. The numbers on the dashed lines denote the dynamic states (MEWs) that correspond to different values of the potential (see below). A similar plot for $k_{u2} < 0$ is obtained by inversion of the plot of Fig. 1 relative to the origin of coordinates.

Type of MEW is determined by relationship of parameters $\underline{e}$, k_{u1}, k_{u2} and m^*. When $\underline{e}$ is above the potential barriers at $m^* > 0$ and lower at $m^* < 0$, the motion of the quasiparticle is infinite. Vector $\vec{l}$ rotates in the basal plane, so that circular nonlinear MEWs (waves of the rotation of antiferromagnetic vector) are observed. When $\underline{e}$ is located between maxima and minima of the potential, the trajectory of motion is bounded by the potential barriers, which corresponds to periodic nonlinear MEWs. When $\underline{e}$ coincides with extrema of $\underline{u}^{\text{eff}}$, the trajectory either lies on the extremum (homogeneous state) or passes from one extre-

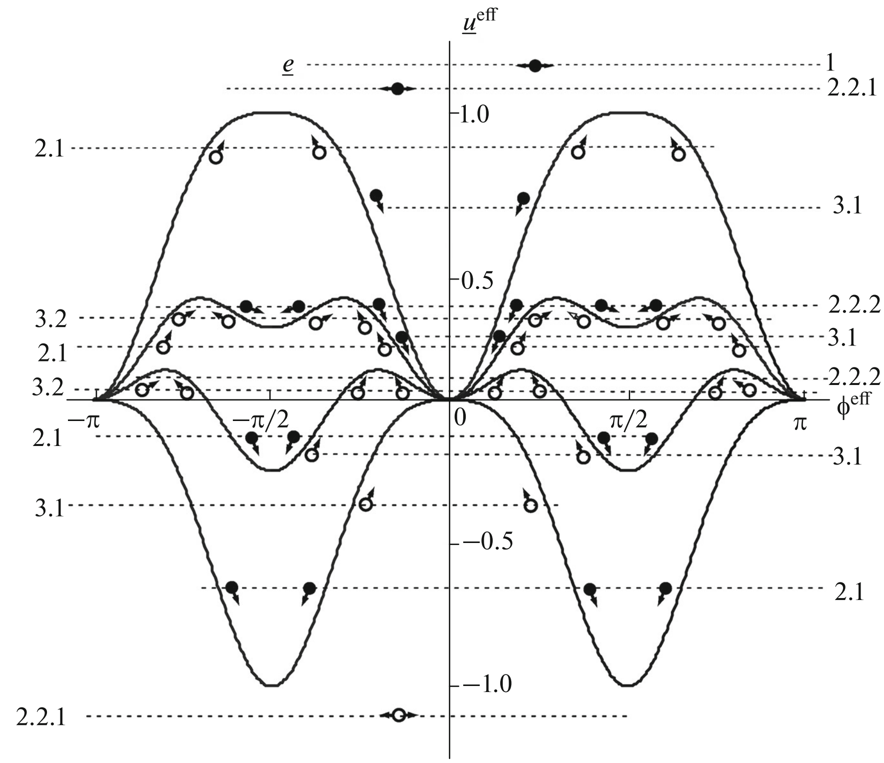

Fig. 1. Plots of dependence $\underline{u}^{\text{eff}}(\phi^{(+)})$: (closed circles) quasiparticles with positive effective mass, (open circles) quasiparticles with negative effective mass, and (arrows) directions of motion in potential $\underline{u}^{\text{eff}}$. The numbers on the dashed lines correspond to the types of dynamic states that exist at different values of the potential.

mum to another (solitary nonlinear MEWs). When asymptotes are located in identical minima of potential, the waves are topologically stable (solitons).

Using the notation $x = \tan\phi^{(+)}$, we represent expression (15) as

$$\int\left[\left(x^2 - x_1^2\right)\left(x^2 - x_2^2\right)\right]^{-1/2} dx = \pm(\xi - \xi_0)\sqrt{\beta}, \quad (16)$$

where

$$x_{1,2}^2 = -\underline{e}_1 \pm \underline{e}_2^{1/2}; \quad \underline{e}_1 = \left(2\underline{e} - k_{u2} + k_{u1}\right)/2\left(\underline{e} + k_{u1}\right)$$
$$= 1 - \left(k_{u1} + k_{u2}\right)/2\left(\underline{e} + k_{u1}\right);$$
$$\beta = 2\left(\underline{e} + k_{u1}\right)/m^*; \quad \underline{e}_0 = \underline{e}/\left(\underline{e} + k_{u1}\right);$$
$$\underline{e}_2 = \underline{e}_1^2 - \underline{e}_0 = -k_{u2}\left(\underline{e} - u_3^{\text{eff}}\right)/\left(\underline{e} - u_2^{\text{eff}}\right)^2.$$

Poles $x_{1,2}^2$ that are complex conjugate at $\underline{e}_2 < 0$, become real at $\underline{e}_2 > 0$. If $\underline{e}_2 < 0$, $x_1^2 > 0$, $x_2^2 < 0$, and $x_1^2 < \left|x_2^2\right|$ for $\underline{e}_2 > 0$, and $x_1^2 > \left|x_2^2\right|$ for $\underline{e}_2 < 0$; if $\underline{e}_2 > 0$, poles $x_{1,2}^2$ are negative at $\underline{e}_1 < 0$ and $\left|x_1^2\right| < \left|x_2^2\right|$, if $\underline{e}_1 < 0$, the poles are positive and $\left|x_1^2\right| > \left|x_2^2\right|$. Using the data for poles, we calculate the integral in expression (16) using the tables of [36] and represent the solution in terms of the Jacobi elliptic functions $\operatorname{sn}(\xi,\kappa)$, $\operatorname{cn}(\xi,\kappa)$, and $\operatorname{dn}(\xi,\kappa)$, where κ is the modulus of function. The solutions can be used to determine the domains of existence and types of possible dynamic states.

Dynamic state 1: $k_{u2}\left(\underline{e} - \underline{u}_3^{\text{eff}}\right) > 0$; $\beta > 0$. In this case, the azimuth angle is calculated as

$$\phi^{(+)} = \arctan\left\{\pm\underline{e}_0^{1/4}\left[\left(1 + \operatorname{cn}(\xi_1,\kappa_1)\right)/\left(1 - \operatorname{cn}(\xi_1,\kappa_1)\right)\right]^{1/2}\right\} + n\pi,$$
$$(n = 0, \pm 1, \ldots), \quad (17)$$

where $\xi_1 = (\xi - \xi_0)/\delta_1$, $\delta_1 = \left(\beta \underline{e}_0^{1/2}\right)^{-1/2}$ is the characteristic length. If $|\underline{e}| > k_{u1}$ and $|\underline{e}| \gg (1/2)|k_{u1} - k_{u2}|$, the modulus is $\kappa_1^2 = k_{u2}/4\underline{e}$; if $\kappa_1 \to 0$ cn $\to$ cos, $\underline{e}_0 \to 1$, so that

$$\phi^{(+)} = (1/2)(\pi \pm \xi_1) + n\pi. \quad (18)$$

In a static coordinate cross section, vector $\vec{l}$ rotates with time in the basal plane. At a fixed time moment, vector $\vec{l}$ rotates with displacement along the x axis, so that we obtain circular nonlinear MEW. At $\underline{e} \to \underline{u}_3^{\text{eff}}$, $\kappa_1 \to 1$, $\text{cn} \to \tanh/\sinh$, and $\underline{e}_0 \to (k_{u2} - k_{u1})^2/(k_{u2} + k_{u1})^2 = \tan^4 \phi_{m3}^{(+)}$, expression (16) describes a solitary nonlinear MEW for which

$$\phi^{(+)} = \operatorname{arccot}\left[\pm \underline{e}_0^{-1/4} \tanh(\xi_1/2)\right] + n\pi. \quad (19)$$

Asymptotic values $\phi^{(+)}(\pm\infty)$ are shifted from $\pi/2$ by $\operatorname{arcctg} \phi_{m3}^{(+)}$.

Substituting expression (17) in expression (11), we find polar angle $\theta_1^{(+)}$ of the exit of antiferromagnetic vector from the basal plane:

$$\begin{gathered}\theta_1^{(+)} = (\pi/2) - \theta^{(+)} \\ = \pm\delta_{v1}\underline{e}_0^{1/4} F_1 \,\text{dn}(\xi_1, \kappa_1)/(1 - \text{cn}(\xi_1, \kappa_1)); \\ \delta_{v1} = \delta_v/\delta_1, \delta_v = -v/2\omega_M k_1, \quad F_1 = \left(1 + a^2\right)^{-1}, \\ a^2 = \underline{e}_0^{1/2}(1 + \text{cn}(\xi_1, \kappa_1))/(1 - \text{cn}(\xi_1, \kappa_1)).\end{gathered} \quad (20)$$

Polar angle $\theta^{(+)}$ periodically changes from minimum $\theta_m^{(+)} = \delta_{v1}\underline{e}_{01} \times \left[\left(1 - \kappa_1^2\right)/\left(1 + \underline{e}_{02}^2\left(\kappa_1^{-2} - 1\right)\right)\right]$ at point $\text{cn}(\xi_1, \kappa_1) = \underline{e}_{02}\left(1 - \kappa_1^{-2}\right)$, where $\underline{e}_{01} = \underline{e}_0^{1/4}/\left(1 + \underline{e}_0^{1/2}\right)$, $\underline{e}_{02} = \left(1 - \underline{e}_0^{1/2}\right)/\left(1 + \underline{e}_0^{1/2}\right)$ to maximum $\theta_M^{(-)} = \delta_{v1}/2\underline{e}_0^{1/4}$ at point $\text{sn}(\xi_1, \kappa_1) = 0$. At $\kappa_1 \to 1$, we have $\theta_M^{(+)} \to 0$. When the direction of rotation of vector $\vec{l}$ changes, the sign of the angle of exit from the basal plane is changed.

Substituting expression (14) in expression (10) with allowance for formula (17), we find

$$u_{xx} = -b_{1v}F_1 a^2, \quad u_{yx} = -b_{6v}F_1 a. \quad (21)$$

Exit of vector $\vec{l}$ from the basal plane is related to the strain

$$u_{zx} = b_{4v}\cos\phi^{(+)}\sin\theta_1^{(+)}, \quad (22)$$

where $b_{4v} = b_4/4\rho\left(v^2 - v_{t2}^2\right)$, $v_{t2}^2 = \lambda_{44}/\rho$, so that we obtain

$$u_{zx} = b_{4v}\delta_{v1}e_0^{1/4}F_1^{3/2}\,\text{dn}(\xi_1, \kappa_1)/(1 - \text{cn}(\xi_1, \kappa_1)). \quad (23)$$

Strain u_{xx} ranges from 0 to b_{1v} and has zero mean value, the amplitude of the alternating strain is $u_{yx0} = b_{6v}$, and parameter $\underline{e}$ is given by $\underline{e} = u_{zx0}^2 m^*/\left(b_{4v}\delta_v\right)^2$, where u_{zx0} is the amplitude of u_{zx}. Nonlinear circular MEWs are excited by the strains determined by expressions (21) and (23), and the domains of existence are given by inequalities

$$\begin{gathered}k_{u2} > 0, \quad \underline{e} > \underline{u}_3^{\text{eff}} > \max\left(0, \underline{u}_2^{\text{eff}}\right), \quad m^* > 0, \\ k_{u2} < 0, \quad \underline{e} < \underline{u}_3^{\text{eff}} < \min\left\{0, \underline{u}_2^{\text{eff}}\right\}, \quad m^* < 0.\end{gathered} \quad (24)$$

Dynamic state 2: $k_{u2}\left(\underline{e} - \underline{u}_3^{\text{eff}}\right) < 0$, $\beta > 0$. The type of state depends on the sign of $\underline{e}_0$: $\underline{e}_0 < 0$ at $0 < \underline{e} < -k_{u1}$ or $-k_{u1} < \underline{e} < 0$; $\underline{e}_0 > 0$ at $\underline{e} > \max\{0, -k_{u1}\}$ or $\underline{e} < \min\{0, -k_{u1}\}$.

Dynamic state 2.1: $\underline{e}_0 < 0$. Equation (16) has two solutions

$$\begin{gathered}\phi^{(+)} = \operatorname{arccot}\left[\pm\left(x_1^2\right)^{-1/2}\text{cn}(\xi_2, \kappa_2)\right] + n\pi, \\ \phi^{(+)} = \operatorname{arccot}\left[\pm\left(x_1^2 - x_2^2\,\text{cn}^2(\xi_2, \kappa_2)\right)^{-1/2}\text{sn}(\xi_2, \kappa_2)\right] + n\pi,\end{gathered} \quad (25)$$

where $\kappa_2^2 = -x_2^2/\left(x_1^2 - x_2^2\right) = \left(1 + \underline{e}_1\underline{e}_2^{-1/2}\right)/2$; $\xi_2 = (\xi - \xi_0)/\delta_2$; $\delta_2 = \left(2\beta \underline{e}_2^{1/2}\right)^{1/2}$ is the characteristic length; $0 \le \kappa_2^2 \le 1/2$ for $\underline{e}_1 < 0$ and $1/2 \le \kappa_2^2 \le 1$ for $\underline{e}_1 > 0$. We have $\kappa_2^2 \to 1/2$, when $\underline{e} \to 2(k_{u2} - k_{u1})$. For $\kappa_{u2} > 0$, the modulus $\kappa_2^2 \to 1/2$, when $\underline{e} \to u_1^{\text{eff}}$. We have $\kappa_2 \to 0$ at $\underline{e} \to -k_{u1}$ in the intervals $k_{u1} < -k_{u2}$, $k_{u1} > 0$ and also at $\underline{e} \to 0$ in the interval $0 < k_{u1} < k_{u2}$. We have $\kappa_2^2 \to 1$ at $\underline{e} \to 0$ in the intervals $k_{u1} > k_{u2}$, $k_{u1} < 0$ and at $\underline{e} \to -k_{u1}$ in the interval $-k_{u2} \le k_{u1} < 0$. For $\kappa_{u2} < 0$, the intervals are obtained using the inversion of the above intervals. Expressions (25) describe the periodic nonlinear MEWs with oscillations of $\vec{l}$ relative to the y axis with the amplitude $\phi_0^{(+)} = \operatorname{arccot}\left(x_1^2\right)^{-1/2}$ that are phase-shifted by one quarter of the period of elliptic functions.

For $\kappa_2 \to 0$, expressions (25) correspond to harmonic waves

$$\phi^{(+)} = \alpha \cos \xi_2, \quad \phi^{(+)} = \alpha \sin \xi_2,$$

where the amplitude is $\alpha = [(\underline{e} + k_{01})/(k_{u2} - k_{u1} - 2\underline{e})]^{1/2} \to 0$ at $\underline{e} \to \underline{u}_2^{\text{eff}}$.

For $\kappa_2 \to 1$, we have $\text{sn} \to \tanh$, $\text{cn} \to \tanh/\sinh$, $x_1^2 \to 0$, and $x_2^2 \to (k_{u2}/k_{u1}) - 1$, so that expressions (21) correspond to either a constant or a solitary wave with asymptotic values of 0 and π:

$$\phi^{(+)} \to 0 \text{ and}$$
$$\phi^{(+)} \to \text{arccot}\left\{\pm[1 - (k_{u2}/k_{u1})]^{-1/2} \sinh\xi_2\right\} + n\pi. \tag{26}$$

Expressions (11) and (25) show that the angle of exit of vector $\vec{l}$ from the basal plane is given by

$$\theta_1^{(+)} = \mp\delta_{v2}\left(x_1^2\right)^{-1/2} F_{21}\,\text{sn}\,(\xi_2,\kappa_2)\,\text{d}\,\text{n}\,(\xi_2,\kappa_2),$$
$$\theta_1^{(+)} = \mp\delta_{v2}\left(x_1^2 - x_2^2\right) F_{22}F_{23}\,\text{c}\,\text{n}\,(\xi_2,\kappa_2)\,\text{d}\,\text{n}\,(\xi_2,\kappa_2), \tag{27}$$

where

$$F_{21} = \left[1 + x_1^2\text{cn}^2(\xi_2,\kappa_2)\right]^{-1},$$
$$F_{22} = \left[1 + F_{23}\text{sn}^2(\xi_2,\kappa_2)\right]^{-1},$$
$$F_{23} = \left[x_1^2 - x_2^2\,\text{cn}^2(\xi_2,\kappa_2)\right]^{-1}.$$

Substituting expressions (25) and (27) in expressions (13) and (22), we find that elastic strains corresponding to the two solutions are represented as

$$u_{xx} = -b_{1v}F_{21}, \quad u_{yx} = \pm b_{6v}F_{21}\,\text{cn}\,(\xi_2,\kappa_2),$$
$$u_{zx} = \pm b_{4v}\delta_{v2}\left(x_1^2\right)^{-1}$$
$$\times F_{21}^{3/2}\,\text{sn}\,(\xi_2,\kappa_2)\,\text{cn}\,(\xi_2,\kappa_2)\,\text{dn}\,(\xi_2,\kappa_2),$$
$$u_{xx} = -b_{1v}F_{22}, \quad u_{yx} = \pm b_{6v}F_{22}F_{23}^{1/2}\text{sn}\,(\xi_2,\kappa_2), \tag{28}$$
$$u_{zx} = \pm b_{4v}\delta_{v2}\left(x_1^2 - x_2^2\right)^{-1} F_{22}^{3/2}F_{23}^2 \times$$
$$\times\,\text{sn}\,(\xi_2,\kappa_2)\,\text{c}\,\text{n}\,(\xi_2,\kappa_2)\,\text{d}\,\text{n}\,(\xi_2,\kappa_2).$$

Mean values of u_{xx} differ from zero whereas quantities u_{yx} and u_{zx} averaged over period are zeros, and parameter $\underline{e}$ is represented in terms of the wave amplitudes. The solutions under consideration exist when

$$k_{u2} > 0; \; -k_{u1} < \underline{e} < 0, \; m^* > 0;$$
$$0 < \underline{e} < -k_{u1}, \; m^* < 0. \tag{29}$$

The type of motion is determined by functional dependence $u^{\text{eff}}\left(\phi^{(+)}\right)$. In the vicinity of the extrema, we obtain nonlinear wave perturbations (cnoidal waves). When the amplitude decreases (increases), the nonlinear waves are transformed into linear harmonic (solitary) waves. Conditions (29) correspond to stable dynamic states. In the presence of dissipation, the oscillations are damped and the magnetic subsystem asymptotically tends to the ground state. The oscillations of quasiparticles with the opposite sign of the effective mass are unstable. However, topological stability (26) is implemented when asymptotic values $\phi^{(+)}(\xi_2)$ correspond to identical extrema separated by a potential barrier. In the presence of dissipation, such topological solitons (that determine the boundary between the vacuum states in the static regime) are slowed down in the absence of shape changes.

Dynamic state 2.2: $e_0 > 0$. The type of motion depends on the sign of $\underline{e}_1$, which is positive at $\underline{e} > -k_{u1}$, $\frac{1}{2}(k_{u2} - k_{u1})$ or $\underline{e} < -k_{u1}$, $\frac{1}{2}(k_{u2} - k_{u1})$ and negative at $k_{u1} > -k_{u2}$, $-k_{u1} < \underline{e} < \frac{1}{2}(k_{u2} - k_{u1})$, or $k_{u1} < -k_{u2}$, $\frac{1}{2}(k_{u2} - k_{u1}) < \underline{e} < -k_{u1}$.

Dynamic state 2.2.1: $\underline{e}_1 > 0$. Two solutions exist:

$$\phi^{(+)} = \arctan\left[\pm\left(-x_1^2\right)^{1/2}\text{sn}\,(\xi_3,\kappa_3)\big/\text{cn}\,(\xi_3,\kappa_3)\right] + n\pi,$$
$$\phi^{(+)} = \arctan\left[\pm\left(-x_2^2\right)^{1/2}\text{cn}\,(\xi_3,\kappa_3)\big/\text{sn}\,(\xi_3,\kappa_3)\right] + n\pi, \tag{30}$$

where $\kappa_3^2 = \left(x_1^2 - x_2^2\right)/\left(-x_2^2\right) = 2\left(1 + \underline{e}_1\underline{e}_2^{-1/2}\right)$ $\delta_3 = \left(-\beta x_2^2\right)^{-1/2}$ and $\xi_3 = (\xi - \xi_0)/\delta_3$ is the characteristic length. Formulas (30) describe the nonlinear MEWs of the rotation of antiferromagnetic vector shifted by $\pi/2$. The following expressions are valid for polar angles:

$$\theta_1^{(+)} = \pm\delta_{v3}\left(-x_1^2\right)^{1/2} F_{31}\,\text{dn}\,(\xi_3,\kappa_3),$$
$$F_{31} = \text{cn}^2(\xi_3,\kappa_3) + x_1^2\,\text{sn}^2(\xi_3,\kappa_3),$$
$$\theta_1^{(+)} = \pm\delta_{v3}\left(-x_2^2\right)^{1/2} F_{32}\,\text{dn}\,(\xi_3,\kappa_3), \tag{31}$$
$$F_{32} = \text{sn}^2(\xi_3,\kappa_3) + x_2^2\,\text{cn}^2(\xi_3,\kappa_3).$$

Vector $\vec{l}$ rotates around the z axis with oscillations with respect to $\theta_1^{(+)}$; extrema of $\theta_1^{(+)}$ correspond to

$$\text{sn}\,(\zeta_3\kappa_3) = 0, \quad \text{cn}\,(\zeta_3\kappa_3) = 0,$$
$$\text{dn}^2(\zeta_3\kappa_3) = \left(\kappa_3^2\big/\left(1 + x_1^2\right)\right) - 1.$$

Expressions for elastic strains are written as

$$u_{xx} = -b_{1v}x_1^2F_{31}\,\text{sn}^2(\xi_3,\kappa_3),$$
$$u_{yx} = -\pm\, b_{6v}\left(-x_1^2\right)F_{31}\,\text{sn}\,(\xi_3,\kappa_3)\,\text{cn}\,(\xi_3,\kappa_3),$$
$$u_{zx} = \mp b_{4v}\delta_{v3}\left(-x_1^2\right)^{1/2} F_{31}^{3/2}\,\text{cn}\,(\xi_3,\kappa_3)\,\text{dn}\,(\xi_3,\kappa_3),$$
$$u_{xx} = b_{1v}x_2^2F_{32}\,\text{cn}^2(\xi_3,\kappa_3), \tag{32}$$
$$u_{yx} = \pm b_{6v}\left(-x_2^2\right)^{1/2} F_{32}\,\text{sn}\,(\xi_3,\kappa_3)\,\text{cn}\,(\xi_3,\kappa_3),$$
$$u_{zx} = \pm b_{4v}\delta_{v3}\left(-x_2^2\right)^{1/2} F_{32}^{3/2}\,\text{sn}\,(\xi_3,\kappa_3)\,\text{dn}\,(\xi_3,\kappa_3).$$

The domain of existence of the above solutions is bounded by the following conditions

$$\begin{aligned} &k_{u2} > 0: \underline{u}_2^{\text{eff}} < \underline{e} < \underline{u}_3^{\text{eff}},\ m^* > 0 \\ &\text{or}\quad \underline{e} < \min\left\{\underline{u}_1^{\text{eff}}, \underline{u}_2^{\text{eff}}\right\},\ m^* < 0; \\ &k_{u2} < 0: \underline{e} > \max\left\{\underline{u}_1^{\text{eff}}, \underline{u}_2^{\text{eff}}\right\},\ m^* > 0 \\ &\text{or}\quad \underline{u}_3^{\text{eff}} < \underline{e} < \underline{u}_2^{\text{eff}},\ m^* < 0. \end{aligned} \tag{33}$$

The quasiparticle moves above (under) barriers at $m^* > 0$ ($m^* < 0$).

Dynamic state 2.2.2: $\underline{e}_1 < 0$. Two solutions exist:

$$\begin{aligned} \phi^{(+)} &= \operatorname{arccot}\left[\pm\left(x_1^2\right)^{-1/2} \operatorname{sn}(\xi_4, \kappa_4)\right] + n\pi, \\ \phi^{(+)} &= \operatorname{arccot}\left[\pm\left(x_1^2 + x_2^2 \operatorname{sn}^2(\xi_4, \kappa_4)\right)^{-1/2}\right] + n\pi, \end{aligned} \tag{34}$$

where $\kappa_4^2 = x_2^2/x_1^2 = \left(\underline{e}_1\underline{e}_2^{-1/2} + 1\right)/\left(\underline{e}_1\underline{e}_2^{-1/2} - 1\right)$; $\xi_4 = (\xi - \xi_0)/\delta_4$; and $\delta_4 = \left(\beta x_1^2\right)^{-1/2}$ is the characteristic length. Formulas (34) describe oscillations with amplitude $\arctan\left(x_1^2\right)^{-1/2}$ with respect to azimuth, and oscillations of $\vec{l}$ relative to the basal plane are given by

$$\begin{aligned} \theta_1^{(+)} &= \pm\delta_{v4}\left(x_1^2\right)^{-1/2} F_{41} \operatorname{cn}(\xi_4, \kappa_4) \operatorname{dn}(\xi_4, \kappa_4), \\ \theta_1^{(+)} &= -\pm\delta_{v4}\left(x_1^2 + x_2^2\right) F_{42} F_{43}^{4/3} \operatorname{sn}(\xi_4, \kappa_4) \operatorname{dn}(\xi_4, \kappa_4), \end{aligned} \tag{35}$$

where $F_{41} = \left[1 + (x_1^2)^{-1} \operatorname{sn}^2(\xi_4, \kappa_4)\right]^{-1}$, $F_{42} = \left[1 + (x_2^2)^{-1} \operatorname{sn}^2(\xi_4, \kappa_4)\right]^{-1}$, and $F_{43} = \left[x_1^2 + x_2^2 \operatorname{sn}^2(\xi_3, \kappa_3)\right]^{-1}$.

The 2.2.2-type oscillations are phase-shifted relative to the 2.1-type oscillations, and the corresponding strains are represented as

$$\begin{aligned} &u_{xx} = -b_{1v} F_{41},\quad u_{yx} = \pm b_{6v}\left(x_1^2\right)^{-1/2} F_{41} \operatorname{sn}(\xi_4, \kappa_4), \\ &u_{zx} = \mp b_{4v}\delta_{v4}\left(x_1^2\right)^{-1} F_{41}^{3/2} \operatorname{sn}(\xi_4, \kappa_4) \\ &\quad\times \operatorname{cn}(\xi_4, \kappa_4) \operatorname{dn}(\xi_4, \kappa_4), \\ &u_{xx} = -b_{1v} F_{42},\ u_{yx} = \pm b_{6v} F_{42} F_{43}^{1/2} \operatorname{cn}(\xi_4, \kappa_4), \\ &u_{zx} = \pm b_{4v}\delta_{v4}\left(x_1^2 + x_2^2\right) F_{42}^{3/2} F_{43}^2 \operatorname{sn}(\xi_4, \kappa_4) \\ &\quad\times \operatorname{cn}(\xi_4, \kappa_4) \operatorname{dn}(\xi_4, \kappa_4). \end{aligned} \tag{36}$$

Solutions (34) are supplemented with the solutions

$$\begin{aligned} \phi^{(+)} &= \arctan\left[\pm\left(x_2^2\right)^{1/2} \operatorname{sn}(\xi_4, \kappa_4)\right], \\ \phi^{(+)} &= \arctan\left[\pm\left(x_2^2\right)^{1/2} \operatorname{cn}(\xi_4, \kappa_4)/\operatorname{dn}(\xi_4, \kappa_4)\right] + n\pi. \end{aligned} \tag{37}$$

Expressions (37) describe oscillations relative to the ground state with the amplitude $\arctan\left(x_2^2\right)^{1/2}$. At $\kappa \to 0$, nonlinear waves (37) are transformed into harmonic waves

$$\phi^{(+)} = \left(x_2^2\right)^{1/2} \sin\xi_4,\quad \phi^{(+)} = \left(x_2^2\right)^{1/2} \cos\xi_4. \tag{38}$$

Note that $x_2^2 \to -\underline{e}/2\left(\underline{e} - 2\left(k_{u2} - k_{u1}\right)\right) \to 0$, when $\underline{e} \to 0$. The parameter is given by

$$\underline{e} = \tan^2\phi_a^{(+)}\left[\tan^2\phi_a^{(+)} - 2\left(k_{u2} - k_{u1}\right)\right]/\left(1 + \tan^2\phi_a^{(+)}\right)^2,$$

where $\phi_a^{(+)}$ is the oscillation amplitude. When $\kappa \to 1$ or $\underline{e} \to \underline{u}^{\text{eff}}$, expressions (37) are transformed into expressions

$$\begin{aligned} \phi^{(+)} &= \arctan\left[\pm\eta \tan\phi_{m3}^{(+)} \tanh(\xi_4)\right] + n\pi, \\ \phi^{(+)} &= \phi_{m3}^{(+)} + n\pi. \end{aligned} \tag{39}$$

Here, the first expression describes a solitary MEW with asymptotic values $\pm\phi_{m3}^{(+)}$. The polar angles that correspond to solution (37) are given by

$$\begin{aligned} \theta_1^{(+)} &= \pm\delta_{v4}\left(x_2^2\right)^{1/2} F_{44} \operatorname{cn}(\xi_4, \kappa_4) \operatorname{dn}(\xi_4, \kappa_4), \\ \theta_1^{(+)} &= \mp\delta_{v4}\left(x_2^2\right)^{1/2}\left(\kappa_4^2 - 1\right) \\ &\quad\times F_{45}^{3/2} \operatorname{sn}(\xi_4, \kappa_4)/\operatorname{dn}^2(\xi_4, \kappa_4), \\ F_{44} &= \left[1 + x_2^2 \operatorname{sn}^2(\xi_4, \kappa_4)\right]^{-1}, \\ F_{45} &= \left[1 + x_2^2 \operatorname{cn}^2(\xi_4, \kappa_4) \operatorname{dn}^{-2}(\xi_4, \kappa_4)\right]^{-1}. \end{aligned} \tag{40}$$

Maximum values of $\theta_1^{(+)}$ are reached at $\operatorname{sn} = 0$ and $\operatorname{cn} = 0$. The strains are represented as

$$\begin{aligned} u_{xx} &= -b_{1v} x_2^2 F_{44} \operatorname{sn}^2(\xi_4, \kappa_4), \\ u_{yx} &= b_{6v}\left(x_2^2\right)^{1/2} F_{44} \operatorname{sn}(\xi_4, \kappa_4), \\ u_{zx} &= \mp b_{4v}\delta_{v4}\left(x_2^2\right)^{1/2} F_{44}^{3/2} \operatorname{cn}(\xi_4, \kappa_4) \operatorname{dn}(\xi_4, \kappa_4), \\ u_{xx} &= -b_{1v} x_2^2 F_{45} \operatorname{cn}^2(\xi_4, \kappa_4), \\ u_{yx} &= b_{6v}\left(x_2^2\right)^{1/2} F_{45} \operatorname{cn}(\xi_4, \kappa_4)/\operatorname{dn}(\xi_4, \kappa_4), \\ u_{zx} &= -b_{4v}\delta_{v4}\left(x_2^2\right)^{1/2}\left(\kappa_4^2 - 1\right) F_{45}^{3/2} \\ &\quad\times \operatorname{sn}(\xi_4, \kappa_4)/\operatorname{dn}^2(\xi_4, \kappa_4). \end{aligned} \tag{41}$$

The domain of existence of such solutions is represented as

$$\begin{aligned} &k_{u2} > \left|k_{u1}\right|,\quad \max\{0, -k_{u1}\} < \underline{e} < \underline{u}_3^{\text{eff}},\quad m^* > 0; \\ &k_{u2} < \left|k_{u1}\right|,\quad \underline{u}_3^{\text{eff}} < \underline{e} < \min\{0, -k_{u1}\}, \\ &m^* < 0. \end{aligned} \tag{42}$$

Dynamic state 3: $k_{u2}\left(\underline{e} - \underline{u}_3^{\text{eff}}\right) < 0,\ \beta < 0$. In this case, the result depends on the sign of $\underline{e}_0$.

Dynamic state 3.1: $\underline{e}_0 < 0$. Two solutions exist:

$$\phi^{(+)} = \arctan\left[\pm\left(x_1^2\right)^{1/2}\operatorname{cn}(\xi_5,\kappa_5)\right] + n\pi,$$
$$\phi^{(+)} = \arctan\left[\pm\left(x_2^2\right)^{1/2}\kappa_5\operatorname{sn}(\xi_5,\kappa_5)/\operatorname{dn}(\xi_5,\kappa_5)\right] + n\pi, \tag{43}$$

where $\xi_5 = (\xi - \xi_0)/\delta_5$, $\delta_5 = \left[\beta\left(x_1^2 + x_2^2\right)\right]^{-1/2}$, $\kappa_5^2 = x_1^2/\left(x_1^2 + x_2^2\right) = \left(1 - \underline{e}_1^{-1}\underline{e}_2^{1/2}\right)/2$.

Oscillations (43) are phase-shifted relative to solutions (37), and the polar angles are given by

$$\begin{aligned}\theta_1^{(+)} &= \mp\delta_{v5}\left(x_1^2\right)^{1/2} F_{51}\operatorname{sn}(\xi_5,\kappa_5)\operatorname{dn}(\xi_5,\kappa_5),\\ \theta_1^{(+)} &= \pm\delta_{v5}\left(x_2^2\right)^{1/2} F_{52}\operatorname{cn}(\xi_5,\kappa_5)/\operatorname{dn}^2(\xi_5,\kappa_5),\\ F_{51} &= \left[1 + x_1^2\operatorname{cn}^2(\xi_5,\kappa_5)\right]^{-1},\\ F_{52} &= \left[1 + x_2^2\kappa_5^2\operatorname{sn}^2(\xi_5,\kappa_5)\operatorname{dn}^{-2}(\xi_5,\kappa_5)\right]^{-1}.\end{aligned} \tag{44}$$

Strains in MEW are described using the relationships

$$\begin{aligned}u_{xx} &= -b_{1v}x_1^2F_{51}\operatorname{cn}^2(\xi_5,\kappa_5),\\ u_{yx} &= b_{6v}\left(x_1^2\right)^{1/2}F_{51}\operatorname{cn}(\xi_4,\kappa_5),\\ u_{zx} &= b_{4v}\delta_{v5}\left(x_1^2\right)^{1/2}F_{51}^{3/2}\operatorname{sn}(\xi_5,\kappa_5)\operatorname{dn}(\xi_5,\kappa_5),\\ u_{xx} &= -b_{1v}x_2^2\kappa_5^2F_{52}\operatorname{sn}^2(\xi_5,\kappa_5)/\operatorname{dn}^2(\xi_5,\kappa_5),\\ u_{yx} &= b_{6v}\left(x_2^2\right)^{1/2}\kappa_5F_{52}\operatorname{sn}(\xi_5,\kappa_5)/\operatorname{dn}(\xi_5,\kappa_5),\\ u_{zx} &= -b_{4v}\delta_{v5}\left(x_2^2\right)^{1/2}\kappa_5F_{52}^{3/2}\operatorname{cn}(\xi_5,\kappa_5)/\operatorname{dn}^2(\xi_5,\kappa_5),\end{aligned} \tag{45}$$

and the domain of existence of such solutions is represented as

$$0 < \underline{e} < -k_{u1},\ m^* > 0;\ -k_{u1} < \underline{e} < 0,\ m^* < 0. \tag{46}$$

Dynamic states 3.2: $\underline{e}_0 > 0$. Two solutions exist:

$$\begin{aligned}\phi^{(+)} &= \arctan\left[\pm\left(x_1^2\right)^{1/2}\operatorname{dn}(\xi_6,\kappa_6)\right] + n\pi,\\ \phi^{(+)} &= \operatorname{arccot}\left[\pm\left(x_2^2\right)^{-1/2}\operatorname{dn}(\xi_6,\kappa_6)\right] + n\pi,\end{aligned} \tag{47}$$

where $\xi_6 = (\xi - \xi_0)/\delta_6$, $\delta_6 = \left(\beta x_1^2\right)^{-1/2}$, $\kappa_6^2 = \left(x_1^2 - x_2^2\right)/x_1^2 = 2/\left(1 - \underline{e}_1^{-1}\underline{e}_2^{1/2}\right)$, and $\underline{e}_1 < 0$. Expressions (47) describe a periodic nonlinear MEW. At $\kappa_6 \to 0$, the oscillations are localized in the vicinity of $\phi_{m3}^{(+)}$. The expressions for polar angles are written as

$$\begin{aligned}\theta_1^{(+)} &= \mp\delta_{v6}\left(x_1^2\right)^{1/2}\kappa_6^2F_{61}\operatorname{sn}(\xi_6,\kappa_6)\operatorname{cn}(\xi_6,\kappa_6),\\ \theta_1^{(+)} &= \mp\delta_{v6}\left(x_2^2\right)^{-1/2}\kappa_6^2F_{62}\operatorname{sn}(\xi_6,\kappa_6)\operatorname{cn}(\xi_6,\kappa_6),\end{aligned} \tag{48}$$

where $F_{61} = \left[1 + x_1^2\operatorname{dn}^2(\xi_6,\kappa_6)\right]^{-1}$, $F_{62} = \left[1 + \left(x_2^2\right)^{-1}\operatorname{dn}^2(\xi_6,\kappa_6)\right]^{-1}$, and the strains are represented as

$$\begin{aligned}u_{xx} &= -b_{1v}x_1^2F_{61}\operatorname{dn}^2(\xi_6,\kappa_6),\\ u_{yx} &= b_{6v}\left(x_1^2\right)^{1/2}F_{61}\operatorname{dn}(\xi_6,\kappa_6),\\ u_{zx} &= b_{4v}\delta_{v6}\left(x_1^2\right)^{1/2}\kappa_6^2F_{61}^{3/2}\operatorname{sn}(\xi_6,\kappa_6)\operatorname{cn}(\xi_6,\kappa_6),\\ u_{xx} &= -b_{1v}F_{62},\quad u_{yx} = b_{6v}\left(x_2^2\right)^{-1/2}F_{62}\operatorname{dn}(\xi_6,\kappa_6),\\ u_{zx} &= -b_{4v}\delta_{v6}\left(x_2^2\right)^{-1}\kappa_6^2F_{62}^{3/2}\\ &\times\operatorname{sn}(\xi_6,\kappa_6)\operatorname{cn}(\xi_6,\kappa_6)\operatorname{dn}(\xi_6,\kappa_6).\end{aligned} \tag{49}$$

The domain of existence of such solutions is described using the expressions

$$\begin{aligned}&\kappa_{u2} > |\kappa_{u1}|,\ \max\{0,-\kappa_{u1}\} < \underline{e} < \underline{u}_3^{\text{eff}},\quad m^* < 0;\\ &\kappa_{u2} < |\kappa_{u1}|,\ \underline{u}_3^{\text{eff}} < \underline{e} < \min\{0,-\kappa_{u1}\},\quad m^* > 0.\end{aligned} \tag{50}$$

The quasiparticle moves in the vicinity of maximum $\underline{u}_3^{\text{eff}}$ at $m^* < 0$ and in the vicinity of minimum $\underline{u}_3^{\text{eff}}$ at $m^* > 0$.

Figure 2 shows the diagram of dynamic states on dimensionless coordinates $(\underline{e}^*, k_{u1}^*)$, where $\underline{e}^* = e/k_{u2}$, $k_{u1}^* = k_{u1}/k_{u2}$ for $k_{u2} > 0$. A similar diagram for $k_{u2} < 0$ is obtained using the inversion of the diagram of Fig. 2 relative to the origin of coordinates.

Each of the above dynamic states is characterized by certain symmetry. In particular, in the periodic nonlinear MEW, quantity $\theta_1^{(+)}$ is symmetric relative to the basal plane and the result of averaging over the period is zero. For the nonlinear MEW of the rotation of antiferromagnetic vector, such a mean value differs from zero. For the phase transition between such waves, the minimum (with respect to modulus) value of $\theta_{1m}^{(+)}$ may serve as the order parameter. Such a value is zero for the periodic wave and proportional to $\left(1 - \kappa_1^2\right)^{1/2}$ for the wave of rotation of antiferromagnetic vector. The root dependence of order parameter $\theta_{1m}^{(+)}$ is typical of any phase transition. Derivative $\partial\theta_{1m}^{(+)}/\partial\underline{e}$ at $e \to \underline{u}_3^{\text{eff}}$ exhibits a singularity that is similar to that of the susceptibility of ferromagnetic materials at the Curie point. Hence, the above changing of the types of wave can be considered as a dynamic phase transition.

4. DISPERSION OF NONLINEAR MEWs

Nonlinear MEWs are represented in terms of elliptic functions that have periods $4K(\kappa)$ for sn, cn and $2K(\kappa)$ for dn, where $K(\kappa)$ is the complete elliptic

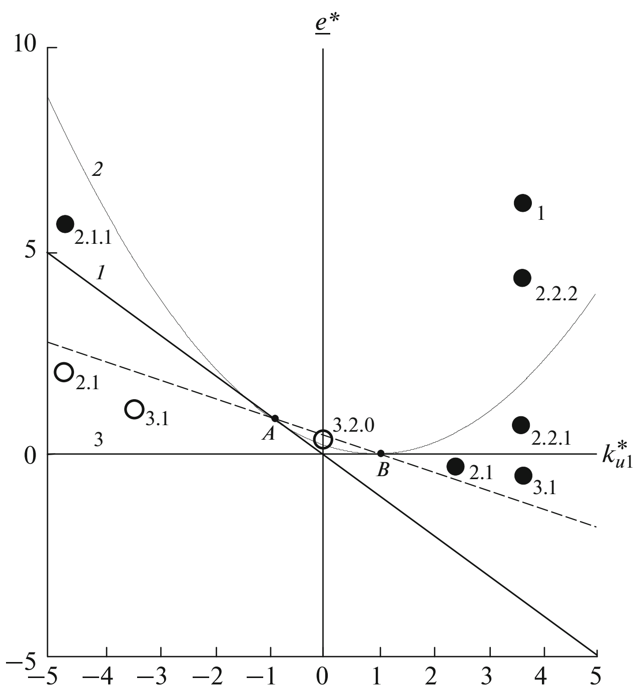

Fig. 2. Diagram of dynamic states on dimensionless coordinates ($\underline{e}^*, k_{u1}^*$) for $k_{u2} > 0$: the closed and open circles denote quasiparticles with positive and negative effective masses, respectively, and the numbers correspond to the types of waves that exist in the regions of the diagram. The coordinates of points A and B are $\{-1, +1\}$ and $\{+1, 0\}$, respectively.

integral of the first kind. When $\kappa \to 0$, integral $K(\kappa)$ tends to $\pi/2$. For $\kappa \to 1$, the integral increases as $K(\kappa) \to \ln\left[4/\left(1-\kappa^2\right)\right]$. The relationship of spatial and time periods is $\Lambda = vT = nK\delta$, where $n = 4$ for sn, cn and $n = 2$ for dn and δ is the characteristic size. With allowance for such a relationship, the expression for the frequency of the 3.1-type wave is written as

$$\omega = (\pi/K)\, v\left[(\kappa_{u2} - \kappa_{u1} - 2\underline{e})/m^*\right]^{1/2} = (r^*/m^*)^{1/2}, \quad (51)$$

where $r^* = (\pi/K)^2 v^2 (k_{u2} - k_{u1} - 2\underline{e})$ is the effective rigidity. Integral K depends on modulus of κ, which depends on v, so that the dispersion relation cannot be represented explicitly. Thus, we consider K as a parameter. For $v = \omega/k$ (where $k = 2\pi/\Lambda$ is the wave number), we obtain

$$\omega_{s,t}^2 = (1/2)\left[\left(s^2 + v_{t1}^2\right)k^2 + \Omega_e^2\right] \pm\left\{(1/4)\left[\left(s^2 - v_{t1}^2\right)k^2 + \Omega_e^2\right]^2 + v_{t1}^2 k^2 \Omega_{me}^2\right\}^{1/2}, \quad (52)$$

where $\Omega_e^2 = (\pi s/K)^2 (k_e/a_{11})$ is the activation energy, $k_e = k_c + k_b - 2\underline{e}$, $k_c = k_{24} - (\sigma_{11} - \sigma_{22})$ is the critical effective constant of anisotropy (at the OPT point $k_c = 0$), $k_b = \left(b_6^2/4\lambda_{66}\right) + \left(b_1^2/(\lambda_{12} - \lambda_{11})\right)$ is the magnetoelastic constant of anisotropy, and $\Omega_{me}^2 = (\pi s/K)^2\left(b_6 b_{66}^*/2a_{11}\right)$ is the parameter that determines the magnetoelastic repulsion of branches. Constant k_e (that determines the activation energy and depends on the wave amplitude) depends only on the magnetoelastic constant at the OPT point. Formula (52) describes the dispersion of coupled spin waves and elastic transverse waves. For $\underline{e} \to 0$ ($|\underline{e}| \ll k_c$), the modulus is $\kappa \to 0$ and the integral is $K \to \pi/2$, $\Omega_e^2 \to 4s^2\left(k_e^2/a_{11}\right)$, $\Omega_e^2 \to 4s^2\left(k_e^2/a_{11}\right)$, $\Omega_{me}^2 \to 2s^2 b_6 b_{66}^*/a_{11}$, $\mathrm{cn}(\xi,\kappa) \to \sin\xi$, $\mathrm{sn}(\xi,\kappa) \to \cos\xi$, $\mathrm{dn}(\xi,\kappa) \to 1$, so that formula (52) is transformed into the dispersion relation of linear waves. In the region of small wave numbers, the dispersion of quasi-spin and quasi-elastic waves is described using the expressions

$$\omega_s^2 = \Omega_e^2 + \left[s^2 + \left(\Omega_{me}^2/\Omega_e^2\right)v_{t1}^2\right]k^2,$$
$$\omega_t^2 = v_{t1}^2\left[1 - \left(\Omega_{me}^2/\Omega_e^2\right)\right]k^2. \quad (53)$$

CONCLUSIONS

We have theoretically analyzed the behavior of nonlinear MEWs in the vicinity of the OPT between the plane collinear and angular phases for a tetragonal antiferromagnetic material. Seven different dynamic states with certain symmetries are possible for the system under study. Transitions between such states may result from changes of parameters of external elastic force.

The motion of antiferromagnetic vector in the presence of the field of elastic wave is similar to the motion of an effective spin quasiparticle in a periodic crystallographic potential that is modified by magnetoelastic static and dynamic interactions. Dynamic state 1 at $\kappa \to 0$ corresponds to a free classical particle, and state 2.2.1 corresponds to a free quantum quasiparticle that moves above the maxima of potential but is sensitive to its variations. A localized spin quasi-particle oscillates relative to the minimum (maximum) of the periodic potential for the positive (negative) effective mass. The potentials with local minima (states 2 and 3 in Fig. 1) correspond to two energy levels, so that effects typical of two-level quantum systems [37] can be observed in the systems under study: in particular, emission of spin waves due to elastic pumping. Nonlinear eigen-MEWs exhibit singularities with respect to velocity that vanish when the exit of the antiferromagnetic vector from the basal plane is taken into account. Similar dynamic magnetic phase transitions with spontaneous violations of symmetry may take place in the presence of a polyharmonic light field [38].

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REFERENCES

1. E. A. Turov and Yu. P. Irkhin, Fiz. Met. Metalloved. (FMM) **3**, 15 (1956).
2. A. I. Akhiezer, V. G. Bar'yakhtar, S. V. Peletminskii, J. Exp. Theor. Phys. **35**, 228 (1958).
3. C. Kittel, Phys. Rev. **110**, 836 (1958).
4. E. G. Spencer and R. Le Craw, Phys. Rev. Lett. **1**, 241 (1958).
5. H. Matthews and R. Le Craw, Phys. Rev. Lett. **8**, 397 (1962).
6. B. Lüthi, Phys. Lett. A **3** (6), 285 (1963).
7. E. G. Rudashevsky and T. A. Shalnikova, in *Proc. 3rd Reg. Conf. on Physics and Techniques of Low Temperatures. Sept., Prague, 1963* (Verlag, Prague, 1964), p. 84.
8. A. Tasaki and S. Iida, J. Phys. Soc. Jpn. **18**, 1148 (1963).
9. A. S. Borovik-Romanov and E. G. Rudashevskii, J. Exp. Theor. Phys. **47**, 2095 (1964).
10. E. A. Turov and V. G. Shavrov, Fiz. Tverd. Tela (Leningrad) **7**, 217 (1965).
11. E. A. Turov and V. G. Shavrov, Usp. Fiz. Nauk **140**, 429 (1983).
12. V. I. Sheglov, Fiz. Tverd. Tela (Leningrad) **14**, 2180 (1972).
13. M. H. Seavay, Solid State Commun. **10** (2), 219 (1972).
14. P. P. Maksimenkov and V. I. Ozhogin, J. Exp. Theor. Phys. **65**, 657 (1973).
15. I. E. Dikshtein, V. V. Tarasenko, and V. G. Shavrov, J. Exp. Theor. Phys. **67**, 816 (1975).
16. I. E. Dikshtein, V. V. Tarasenko, and V. G. Shavrov, Fiz. Tverd. Tela (Leningrad) **19**, 1107 (1977).
17. S. A. Migachev, Kh. G. Bogdanova, and M. I. Kurkin, Phys. Solid State **57**, 38 (2015).
18. V. I. Ozhogin and A. Yu. Lebedev, J. Magn. Magn. Mater. **15–18**, 617 (1980).
19. S. K. Turitsyn and G. E. Fal'kovich, Zh. Eksp. Teor. Fiz. **89,** 258 (1985).
20. A. F. Kabychenkov and V. G. Shavrov, Zh. Eksp. Teor. Fiz. **95**, 580 (1989).
21. V. V. Kiselev and A. P. Tankeev, Fiz. Met. Metalloved. (FMM) **75**, 40 (1993).
22. M. A. Shamsutdinov, A. T. Kharisov, and A. P. Tankeev, Fiz. Met. Metalloved. **85**, 43 (1998).
23. A. T. Kharisov, M. A. Shamsutdinov, and A. P. Tankeev, Fiz. Met. Metalloved. **87**, 5 (1999).
24. A. Ekpekpo, Int. J. Rec. Res. & Appl. Studies (IJRRAS) **6**, 449 (2011).
25. A. F. Kabychenkov, V. G. Shavrov, and A. L. Shevchenko, Fiz. Tverd. Tela (Leningrad) **32**, 2010 (1990).
26. V. I. Ozhogin and V. L. Preobrazhenskii, Usp. Fiz. Nauk **155**, 593 (1988).
27. V. I. Ozhogin and V. L. Preobrazhenskii, Zh. Eksp. Teor. Fiz. **73**, 988 (1977).
28. A. F. Kabychenkov and V. G. Shavrov, Fiz. Tverd. Tela (Leningrad) **28**, 433 (1986).
29. A. F. Kabychenkov and V. G. Shavrov, Fiz. Tverd. Tela (Leningrad) **29**, 202 (1987).
30. M. V. Chetkin and V. V. Lykov, Pis'ma Zh. Eksp. Teor. Fiz. **52**, 863 (1990).
31. A. F. Kabychenkov, V. G. Shavrov, and A. L. Shevchenko, Fiz. Tverd. Tela (Leningrad) **31**, 193 (1989).
32. A. F. Kabychenkov, V. G. Shavrov, and A. L. Shevchenko, Fiz. Tverd. Tela (Leningrad) **32**,1182 (1990).
33. N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Nauka, Moscow, 1984) [in Russian].
34. A. A. Bogush and L. G. Moroz, *Introduction to the Theory of Classical Field* (URSS, Moscow, 2004) [in Russian].
35. K. B. Vlasov and B. Kh. Ishmukhametov, Zh. Eksp. Teor. Fiz. **46**, 201 (1964).
36. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series,* Vol. 2: *Special Functions* (Nauka, Moscow, 2003; Gordon and Breach, New York, 1986).
37. D. N. Klyshko, *Physical Principles of Quantum Electronics* (Nauka, Moscow, 1986) [in Russian].
38. A. F. Kabychenkov, Phys. Solid State **48**, 516 (2006).

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